

# NOTE ON MATH 4010: FUNCTIONAL ANALYSIS

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Throughout this note, all spaces  $X, Y, \dots$  are normed spaces over the field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $B_X := \{x \in X : \|x\| \leq 1\}$  and  $S_X := \{x \in X : \|x\| = 1\}$  denote the closed unit ball and the unit sphere of  $X$  respectively.

## 1. CLASSICAL NORMED SPACES

**Proposition 1.1.** *Let  $X$  be a normed space. Then the following assertions are equivalent.*

(i)  $X$  is a Banach space.

(ii) If a series  $\sum_{n=1}^{\infty} x_n$  is absolutely convergent in  $X$ , i.e.,  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ , implies that the series  $\sum_{n=1}^{\infty} x_n$  converges in the norm.

*Proof.* (i)  $\Rightarrow$  (ii) is obvious.

Now suppose that Part (ii) holds. Let  $(y_n)$  be a Cauchy sequence in  $X$ . It suffices to show that  $(y_n)$  has a convergent subsequence. In fact, by the definition of a Cauchy sequence, there is a subsequence  $(y_{n_k})$  such that  $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k}$  for all  $k = 1, 2, \dots$ . So by the assumption, the series  $\sum_{k=1}^{\infty} (y_{n_{k+1}} - y_{n_k})$  converges in the norm and hence, the sequence  $(y_{n_k})$  is convergent in  $X$ . The proof is finished.  $\square$

Throughout the note, we write a sequence of numbers as a function  $x : \{1, 2, \dots\} \rightarrow \mathbb{K}$ . The following examples are important classes in the study of functional analysis.

**Example 1.2.** *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K}, \lim_{i \rightarrow \infty} |x(i)| = 0\} \text{ and } \ell^\infty := \{(x(i)) : x(i) \in \mathbb{K}, \sup_i |x(i)| < \infty\}.$$

Then  $c_0$  is a subspace of  $\ell^\infty$ . The sup-norm  $\|\cdot\|_\infty$  on  $\ell^\infty$  is defined by  $\|x\|_\infty := \sup_i |x(i)|$  for  $x \in \ell^\infty$ . Then  $\ell^\infty$  is a Banach space and  $(c_0, \|\cdot\|_\infty)$  is a closed subspace of  $\ell^\infty$  (**Check !**) and hence  $c_0$  is also a Banach space too.

Let

$$c_{00} := \{(x(i)) : \text{there are only finitely many } x(i) \text{'s are non-zero}\}.$$

Also,  $c_{00}$  is endowed with the sup-norm defined above. Then  $c_{00}$  is not a Banach space (**Why?**) but it is dense in  $c_0$ , that is,  $\overline{c_{00}} = c_0$  (**Check!**).

**Example 1.3.** *For  $1 \leq p < \infty$ . Put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K}, \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also,  $\ell^p$  is equipped with the norm  $\|x\|_p := \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}}$  for  $x \in \ell^p$ . Then  $\ell^p$  becomes a Banach space under the norm  $\|\cdot\|_p$ .

**Example 1.4.** Let  $X$  be a locally compact Hausdorff space, for example,  $\mathbb{K}$ . Let  $C_0(X)$  be the space of all continuous  $\mathbb{K}$ -valued functions  $f$  on  $X$  which vanish at infinity, that is, for every  $\varepsilon > 0$ , there is a compact subset  $D$  of  $X$  such that  $|f(x)| < \varepsilon$  for all  $x \in X \setminus D$ . Now  $C_0(X)$  is endowed with the sup-norm, that is,

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

for every  $f \in C_0(X)$ . Then  $C_0(X)$  is a Banach space. (Try to prove this fact for the case  $X = \mathbb{R}$ . Just use the knowledge from MATH 2060 !!!)

## 2. FINITE DIMENSIONAL NORMED SPACES

We say that two norms  $\|\cdot\|$  and  $\|\cdot\|'$  on a vector space  $X$  are *equivalent*, write  $\|\cdot\| \sim \|\cdot\|'$ , if there are positive numbers  $c_1$  and  $c_2$  such that  $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$  on  $X$ .

**Example 2.1.** Consider the norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  on  $\ell^1$ . We are going to show that  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are not equivalent. In fact, if we put  $x_n(i) := (1, 1/2, \dots, 1/n, 0, 0, \dots)$  for  $n, i = 1, 2, \dots$ . Then  $x_n \in \ell^1$  for all  $n$ . Notice that  $(x_n)$  is a Cauchy sequence with respect to the norm  $\|\cdot\|_\infty$  but it is not a Cauchy sequence with respect to the norm  $\|\cdot\|_1$ . Hence  $\|\cdot\|_1 \not\sim \|\cdot\|_\infty$  on  $\ell^1$ .

**Proposition 2.2.** All norms on a finite dimensional vector space are equivalent.

*Proof.* Let  $X$  be a finite dimensional vector space and let  $\{e_1, \dots, e_n\}$  be a vector base of  $X$ . For each  $x = \sum_{i=1}^n \alpha_i e_i$  for  $\alpha_i \in \mathbb{K}$ , define  $\|x\|_0 = \sum_{i=1}^n |\alpha_i|$ . Then  $\|\cdot\|_0$  is a norm on  $X$ . The result is obtained by showing that all norms  $\|\cdot\|$  on  $X$  are equivalent to  $\|\cdot\|_0$ .

Notice that for each  $x = \sum_{i=1}^n \alpha_i e_i \in X$ , we have  $\|x\| \leq (\max_{1 \leq i \leq n} \|e_i\|) \|x\|_0$ . It remains to find  $c > 0$  such that  $c\|\cdot\|_0 \leq \|\cdot\|$ . In fact, let  $\mathbb{K}^n$  be equipped with the sup-norm  $\|\cdot\|_\infty$ , that is  $\|(\alpha_1, \dots, \alpha_n)\|_\infty = \max_{1 \leq i \leq n} |\alpha_i|$ . Define a real-valued function  $f$  on the unit sphere  $S_{\mathbb{K}^n}$  of  $\mathbb{K}^n$  by

$$f : (\alpha_1, \dots, \alpha_n) \in S_{\mathbb{K}^n} \mapsto \|\alpha_1 e_1 + \dots + \alpha_n e_n\|.$$

Notice that the map  $f$  is continuous and  $f > 0$ . It is clear that  $S_{\mathbb{K}^n}$  is compact with respect to the sup-norm  $\|\cdot\|_\infty$  on  $\mathbb{K}^n$ . Hence, there is  $c > 0$  such that  $f(\alpha) \geq c > 0$  for all  $\alpha \in S_{\mathbb{K}^n}$ . This gives  $\|x\| \geq c\|x\|_0$  for all  $x \in X$  as desired. The proof is finished.  $\square$

**Corollary 2.3.** We have the following assertions.

- (i) All finite dimensional normed spaces are Banach spaces. Consequently, any finite dimensional subspace of a normed space must be closed.
- (ii) The closed unit ball of any finite dimensional normed space is compact.

*Proof.* Let  $(X, \|\cdot\|)$  be a finite dimensional normed space. With the notation as in the proof of Proposition 2.2 above, we see that  $\|\cdot\|$  must be equivalent to the norm  $\|\cdot\|_0$ . It is clear that  $X$  is complete with respect to the norm  $\|\cdot\|_0$  and so is complete in the original norm  $\|\cdot\|$ . The Part (i) follows.

For Part (ii), it is clear that the compactness of the closed unit ball of  $X$  is equivalent to saying that any closed and bounded subset being compact. Therefore, Part (ii) follows from the simple observation that any closed and bounded subset of  $X$  with respect to the norm  $\|\cdot\|_0$  is compact. The proof is complete.  $\square$

In the rest of this section, we are going to show the converse of Corollary 2.3(ii) also holds. Before this result, we need the following useful result.

**Lemma 2.4. Riesz's Lemma:** Let  $Y$  be a closed proper subspace of a normed space  $X$ . Then for each  $\theta \in (0, 1)$ , there is an element  $x_0 \in S_X$  such that  $d(x_0, Y) := \inf\{\|x_0 - y\| : y \in Y\} \geq \theta$ .

*Proof.* Let  $u \in X - Y$  and  $d := \inf\{\|u - y\| : y \in Y\}$ . Notice that since  $Y$  is closed,  $d > 0$  and hence, we have  $0 < d < \frac{d}{\theta}$  because  $0 < \theta < 1$ . This implies that there is  $y_0 \in Y$  such that  $0 < d \leq \|u - y_0\| < \frac{d}{\theta}$ . Now put  $x_0 := \frac{u - y_0}{\|u - y_0\|} \in S_X$ . We are going to show that  $x_0$  is as desired. Indeed, let  $y \in Y$ . Since  $y_0 + \|u - y_0\|y \in Y$ , we have

$$\|x_0 - y\| = \frac{1}{\|u - y_0\|} \|u - (y_0 + \|u - y_0\|y)\| \geq d/\|u - y_0\| > \theta.$$

So,  $d(x_0, Y) \geq \theta$ . □

**Remark 2.5.** The Riesz's lemma does not hold when  $\theta = 1$ . The following example can be found in the Diestel's interesting book without proof (see [3, Chapter 1 Ex.3(i)]).

Let  $X = \{x \in C([0, 1], \mathbb{R}) : x(0) = 0\}$  and  $Y = \{y \in X : \int_0^1 y(t)dt = 0\}$ . Both  $X$  and  $Y$  are endowed with the sup-norm. Notice that  $Y$  is a closed proper subspace of  $X$ . We are going to show that for any  $x \in S_X$ , there is  $y \in Y$  such that  $\|x - y\|_\infty < 1$ . Thus, the Riesz's Lemma does not hold as  $\theta = 1$  in this case.

In fact, let  $x \in S_X$ . Since  $x(0) = 0$  with  $\|x\|_\infty = 1$ , we can find  $0 < a < 1/4$  such that  $|x(t)| \leq 1/4$  for all  $t \in [0, a]$ .

We fix  $0 < \varepsilon < 1/4$  first. Since  $x$  is uniform continuous on  $[a, 1]$ , we can find a partitions  $a = t_0 < \dots < t_n = 1$  on  $[a, 1]$  such that  $\sup\{|x(t) - x(t')| : t, t' \in [t_{k-1}, t_k]\} < \varepsilon/4$ . First, we are going to define a continuous function  $\phi$  on  $[a, 1]$  such that  $\|\phi - x|_{[a, 1]}\|_\infty < 1$ . In fact, for each  $(x_{k-1}, x_k)$ , if  $\sup\{x(t) : t \in [t_{k-1}, t_k]\} > \varepsilon$ , then we set  $\phi(t) = \varepsilon$ . Also, if  $\inf\{x(t) : t \in [t_{k-1}, t_k]\} < -\varepsilon$ , then we set  $\phi(t) = -\varepsilon$ . From this, one can construct a continuous function  $\phi$  on  $[a, 1]$  as required.

Moreover, we have  $|\phi(x)| < 2\varepsilon$  for all  $x \in [a, 1]$  and hence, we have  $|\int_a^1 \phi(t)dt| \leq 2\varepsilon(1 - a)$ .

On the other hand, as  $|x(t)| < 1/4$  on  $[0, a]$ , so if we choose  $\varepsilon$  small enough such that  $(1 - a)(2\varepsilon) < a/4$ , then we can find a continuous function  $y_1$  on  $[0, a]$  such that  $|y_1(t)| < 1/4$  on  $[0, a]$  with  $y_1(0) = 0; y_1(a) = x(a)$  and  $\int_0^a y_1(t)dt = -\int_a^1 \phi(t)dt$ . Now we define  $y = y_1$  on  $[0, a]$  and  $y = \phi$  on  $[a, 1]$ . Then  $\|y - x\|_\infty < 1$  and  $y \in Y$  is as desired.

**Theorem 2.6.**  *$X$  is a finite dimensional normed space if and only if the closed unit ball  $B_X$  of  $X$  is compact.*

*Proof.* The necessary condition has been shown by Proposition 2.3(ii).

Now assume that  $X$  is of infinite dimension. Fix an element  $x_1 \in S_X$ . Let  $Y_1 = \mathbb{K}x_1$ . Then  $Y_1$  is a proper closed subspace of  $X$ . The Riesz's lemma gives an element  $x_2 \in S_X$  such that  $\|x_1 - x_2\| \geq 1/2$ . Now consider  $Y_2 = \text{span}\{x_1, x_2\}$ . Then  $Y_2$  is a proper closed subspace of  $X$  since  $\dim X = \infty$ . To apply the Riesz's Lemma again, there is  $x_3 \in S_X$  such that  $\|x_3 - x_k\| \geq 1/2$  for  $k = 1, 2$ . To repeat the same step, there is a sequence  $(x_n) \in S_X$  such that  $\|x_m - x_n\| \geq 1/2$  for all  $n \neq m$ . Thus,  $(x_n)$  is a bounded sequence without any convergence subsequence. So,  $B_X$  is not compact. The proof is finished. □

Recall that a metric space  $Z$  is said to be *locally compact* if for any point  $z \in Z$ , there is a compact neighborhood of  $z$ . Theorem 2.6 implies the following corollary immediately.

**Corollary 2.7.** *Let  $X$  be a normed space. Then  $X$  is locally compact if and only if  $\dim X < \infty$ .*

### 3. BOUNDED LINEAR OPERATORS

**Proposition 3.1.** *Let  $T$  be a linear operator from a normed space  $X$  into a normed space  $Y$ . Then the following statements are equivalent.*

- (i)  $T$  is continuous on  $X$ .
- (ii)  $T$  is continuous at  $0 \in X$ .

(iii)  $\sup\{\|Tx\| : x \in B_X\} < \infty$ .

In this case, let  $\|T\| = \sup\{\|Tx\| : x \in B_X\}$  and  $T$  is said to be bounded.

*Proof.* (i)  $\Rightarrow$  (ii) is obvious.

For (ii)  $\Rightarrow$  (i), suppose that  $T$  is continuous at 0. Let  $x_0 \in X$ . Let  $\varepsilon > 0$ . Then there is  $\delta > 0$  such that  $\|Tw\| < \varepsilon$  for all  $w \in X$  with  $\|w\| < \delta$ . Therefore, we have  $\|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon$  for any  $x \in X$  with  $\|x - x_0\| < \delta$ . So, (i) follows.

For (ii)  $\Rightarrow$  (iii), since  $T$  is continuous at 0, there is  $\delta > 0$  such that  $\|Tx\| < 1$  for any  $x \in X$  with  $\|x\| < \delta$ . Now for any  $x \in B_X$  with  $x \neq 0$ , we have  $\|\frac{\delta}{2}x\| < \delta$ . So, we see have  $\|T(\frac{\delta}{2}x)\| < 1$  and hence, we have  $\|Tx\| < 2/\delta$ . So, (iii) follows.

Finally, it remains to show (iii)  $\Rightarrow$  (ii). Notice that by the assumption of (iii), there is  $M > 0$  such that  $\|Tx\| \leq M$  for all  $x \in B_X$ . So, for each  $x \in X$ , we have  $\|Tx\| \leq M\|x\|$ . This implies that  $T$  is continuous at 0. The proof is complete.  $\square$

**Corollary 3.2.** *Let  $T : X \rightarrow Y$  be a bounded linear map. Then we have*

$$\sup\{\|Tx\| : x \in B_X\} = \sup\{\|Tx\| : x \in S_X\} = \inf\{M > 0 : \|Tx\| \leq M\|x\|, \forall x \in X\}.$$

*Proof.* Let  $a = \sup\{\|Tx\| : x \in B_X\}$ ,  $b = \sup\{\|Tx\| : x \in S_X\}$  and  $c = \inf\{M > 0 : \|Tx\| \leq M\|x\|, \forall x \in X\}$ .

It is clear that  $b \leq a$ . Now for each  $x \in B_X$  with  $x \neq 0$ , then we have  $b \geq \|T(x/\|x\|)\| = (1/\|x\|)\|Tx\| \geq \|Tx\|$ . So, we have  $b \geq a$  and thus,  $a = b$ .

Now if  $M > 0$  satisfies  $\|Tx\| \leq M\|x\|$ ,  $\forall x \in X$ , then we have  $\|Tw\| \leq M$  for all  $w \in S_X$ . So, we have  $b \leq M$  for all such  $M$ . So, we have  $b \leq c$ . Finally, it remains to show  $c \leq b$ . Notice that by the definition of  $b$ , we have  $\|Tx\| \leq b\|x\|$  for all  $x \in X$ . So,  $c \leq b$ .  $\square$

**Proposition 3.3.** *Let  $X$  and  $Y$  be normed spaces. Suppose that  $X$  is of finite dimension  $n$ . Then we have the following assertions.*

(i) *Any linear operator from  $X$  into  $Y$  must be bounded.*

(ii) *If  $T_k : X \rightarrow Y$  is a sequence of linear operators such that  $T_k x \rightarrow 0$  for all  $x \in X$ , then  $\|T_k\| \rightarrow 0$ .*

*Proof.* Using Proposition 2.2 and the notation as in the proof, then there is  $c > 0$  such that

$$\sum_{i=1}^n |\alpha_i| \leq c \left\| \sum_{i=1}^n \alpha_i e_i \right\|$$

for all scalars  $\alpha_1, \dots, \alpha_n$ . Therefore, for any linear map  $T$  from  $X$  to  $Y$ , we have

$$\|Tx\| \leq \left( \max_{1 \leq i \leq n} \|Te_i\| \right) c \|x\|$$

for all  $x \in X$ . This gives the assertions (i) and (ii) immediately.  $\square$

**Proposition 3.4.** *Let  $Y$  be a closed subspace of  $X$  and  $X/Y$  be the quotient space. For each element  $x \in X$ , put  $\bar{x} := x + Y \in X/Y$  the corresponding element in  $X/Y$ . Define*

$$(3.1) \quad \|\bar{x}\| = \inf\{\|x + y\| : y \in Y\}.$$

*If we let  $\pi : X \rightarrow X/Y$  be the natural projection, that is  $\pi(x) = \bar{x}$  for all  $x \in X$ , then  $(X/Y, \|\cdot\|)$  is a normed space and  $\pi$  is bounded with  $\|\pi\| \leq 1$ . In particular,  $\|\pi\| = 1$  as  $Y$  is a proper closed subspace.*

*Furthermore, if  $X$  is a Banach space, then so is  $X/Y$ .*

*In this case, we call  $\|\cdot\|$  in (3.1) the quotient norm on  $X/Y$ .*

*Proof.* Notice that since  $Y$  is closed, one can directly check that  $\|\bar{x}\| = 0$  if and only if  $x \in Y$ , that is,  $\bar{x} = \bar{0} \in X/Y$ . It is easy to check the other conditions of the definition of a norm. So,  $X/Y$  is a normed space. Also, it is clear that  $\pi$  is bounded with  $\|\pi\| \leq 1$  by the definition of the quotient norm on  $X/Y$ .

Furthermore, if  $Y \subsetneq X$ , then by using the Riesz's Lemma 2.4, we see that  $\|\pi\| = 1$  at once.

We are going to show the last assertion. Suppose that  $X$  is a Banach space. Let  $(\bar{x}_n)$  be a Cauchy sequence in  $X/Y$ . It suffices to show that  $(\bar{x}_n)$  has a convergent subsequence in  $X/Y$  (**Why?**). Indeed, since  $(\bar{x}_n)$  is a Cauchy sequence, we can find a subsequence  $(\bar{x}_{n_k})$  of  $(\bar{x}_n)$  such that

$$\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| < 1/2^k$$

for all  $k = 1, 2, \dots$ . Then by the definition of quotient norm, there is an element  $y_1 \in Y$  such that  $\|x_{n_2} - x_{n_1} + y_1\| < 1/2$ . Notice that we have,  $\overline{x_{n_1} - y_1} = \bar{x}_{n_1}$  in  $X/Y$ . So, there is  $y_2 \in Y$  such that  $\|x_{n_2} - y_2 - (x_{n_1} - y_1)\| < 1/2$  by the definition of quotient norm again. Also, we have  $\overline{x_{n_2} - y_2} = \bar{x}_{n_2}$ . Then we also have an element  $y_3 \in Y$  such that  $\|x_{n_3} - y_3 - (x_{n_2} - y_2)\| < 1/2^2$ . To repeat the same step, we can obtain a sequence  $(y_k)$  in  $Y$  such that

$$\|x_{n_{k+1}} - y_{k+1} - (x_{n_k} - y_k)\| < 1/2^k$$

for all  $k = 1, 2, \dots$ . Therefore,  $(x_{n_k} - y_k)$  is a Cauchy sequence in  $X$  and thus,  $\lim_k (x_{n_k} - y_k)$  exists in  $X$  while  $X$  is a Banach space. Set  $x = \lim_k (x_{n_k} - y_k)$ . On the other hand, notice that we have  $\pi(x_{n_k} - y_k) = \pi(x_{n_k})$  for all  $k = 1, 2, \dots$ . This tells us that  $\lim_k \pi(x_{n_k}) = \lim_k \pi(x_{n_k} - y_k) = \pi(x) \in X/Y$  since  $\pi$  is bounded. So,  $(\bar{x}_{n_k})$  is a convergent subsequence of  $(\bar{x}_n)$  in  $X/Y$ . The proof is complete.  $\square$

**Corollary 3.5.** *Let  $T : X \rightarrow Y$  be a linear map. Suppose that  $Y$  is of finite dimension. Then  $T$  is bounded if and only if  $\ker T := \{x \in X : Tx = 0\}$ , the kernel of  $T$ , is closed.*

*Proof.* The necessary part is clear.

Now assume that  $\ker T$  is closed. Then by Proposition 3.4,  $X/\ker T$  becomes a normed space. Also, it is known that there is a linear injection  $\tilde{T} : X/\ker T \rightarrow Y$  such that  $T = \tilde{T} \circ \pi$ , where  $\pi : X \rightarrow X/\ker T$  is the natural projection. Since  $\dim Y < \infty$  and  $\tilde{T}$  is injective,  $\dim X/\ker T < \infty$ . This implies that  $\tilde{T}$  is bounded by Proposition 3.3. Hence  $T$  is bounded because  $T = \tilde{T} \circ \pi$  and  $\pi$  is bounded.  $\square$

**Remark 3.6.** The converse of Corollary 3.5 does not hold when  $Y$  is of infinite dimension. For example, let  $X := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x(n)|^2 < \infty\}$  (notice that  $X$  is a vector space **Why?**) and  $Y = \ell^2$ . Both  $X$  and  $Y$  are endowed with  $\|\cdot\|_2$ -norm.

Define  $T : X \rightarrow Y$  by  $Tx(n) = nx(n)$  for  $x \in X$  and  $n = 1, 2, \dots$ . Then  $T$  is an unbounded operator (**Check !!**). Notice that  $\ker T = \{0\}$  and hence,  $\ker T$  is closed. So, the closeness of  $\ker T$  does not imply the boundedness of  $T$  in general.

We say that two normed spaces  $X$  and  $Y$  are said to be *isomorphic* (resp. *isometric isomorphic*) if there is a bi-continuous linear isomorphism (resp. isometric) between  $X$  and  $Y$ . We also write  $X = Y$  if  $X$  and  $Y$  are isometric isomorphic.

**Remark 3.7.** *Notice that the inverse of a bounded linear isomorphism may not be bounded.*

**Example 3.8.** *Let  $X := \{f \in C^\infty(-1, 1) : f^{(n)} \in C^b(-1, 1) \text{ for all } n = 0, 1, 2, \dots\}$  and  $Y := \{f \in X : f(0) = 0\}$ . Also,  $X$  and  $Y$  both are equipped with the sup-norm  $\|\cdot\|_\infty$ . Define an operator  $S : X \rightarrow Y$  by*

$$Sf(x) := \int_0^x f(t) dt$$

for  $f \in X$  and  $x \in (-1, 1)$ . Then  $S$  is a bounded linear isomorphism but its inverse  $S^{-1}$  is unbounded. In fact, the inverse  $S^{-1} : Y \rightarrow X$  is given by

$$S^{-1}g := g'$$

for  $g \in Y$ .

Recall that a metric space is said to be *separable* if there is a countable dense subset, for example, the base field  $\mathbb{K}$  is separable. Also, it is easy to see that a normed space is separable if and only if it is the closed linear span of a countable dense subset.

**Definition 3.9.** We say that a sequence of element  $(e_n)_{n=1}^{\infty}$  in a normed space  $X$  is called a *Schauder base* for  $X$  if for each element  $x \in X$ , there is a unique sequence of scalars  $(\alpha_n)$  such that

$$(3.2) \quad x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

**Note:** The expression in Eq. 3.2 depends on the order of  $e_n$ 's.

**Remark 3.10.** Notice that if  $X$  has a Schauder base, then  $X$  must be separable. The following natural question we first raised by Banach (1932).

**The base problem:** Does every separable Banach space have a Schauder base?

The answer is “**No**”!

This problem was completely solved by P. Enflo in 1973.

**Example 3.11.** We have the following assertions.

(i) The space  $\ell^{\infty}$  is non-separable under the sup-norm  $\|\cdot\|_{\infty}$ . Consequently,  $\ell^{\infty}$  has no Schauder base.

(ii) The spaces  $c_0$  and  $\ell^p$  for  $1 \leq p < \infty$  have Schauder bases.

*Proof.* For Part (i) let  $D = \{x \in \ell^{\infty} : x(i) = 0 \text{ or } 1\}$ . Then  $D$  is an uncountable set and  $\|x - y\|_{\infty} = 1$  for  $x \neq y$ . Therefore  $\{B(x, 1/4) : x \in D\}$  is an uncountable family of disjoint open balls. So,  $\ell^{\infty}$  has no countable dense subset.

For each  $n = 1, 2, \dots$ , let  $e_n(i) = 1$  if  $n = i$ , otherwise, is equal to 0.

Also,  $(e_n)$  is a Schauder base for the space  $c_0$  and  $\ell^p$  for  $1 \leq p < \infty$ . □

**Proposition 3.12.** Let  $X$  and  $Y$  be normed spaces. Let  $B(X, Y)$  be the set of all bounded linear maps from  $X$  into  $Y$ . For each element  $T \in B(X, Y)$ , let

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

be defined as in Proposition 3.1.

Then  $(B(X, Y), \|\cdot\|)$  becomes a normed space.

Furthermore, if  $Y$  is a Banach space, then so is  $B(X, Y)$ .

*Proof.* One can directly check that  $B(X, Y)$  is a normed space (**Do It By Yourself!**).

We are going to show that  $B(X, Y)$  is complete if  $Y$  is a Banach space. Let  $(T_n)$  be a Cauchy sequence in  $L(X, Y)$ . Then for each  $x \in X$ , it is easy to see that  $(T_n x)$  is also a Cauchy sequence in  $Y$ . So,  $\lim T_n x$  exists in  $Y$  for each  $x \in X$  because  $Y$  is complete. Hence, one can define a map  $Tx := \lim T_n x \in Y$  for each  $x \in X$ . It is clear that  $T$  is a linear map from  $X$  into  $Y$ .

It needs to show that  $T \in L(X, Y)$  and  $\|T - T_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\varepsilon > 0$ . Since  $(T_n)$  is a Cauchy sequence in  $L(X, Y)$ , there is a positive integer  $N$  such that  $\|T_m - T_n\| < \varepsilon$  for all  $m, n \geq N$ . So, we have  $\|(T_m - T_n)(x)\| < \varepsilon$  for all  $x \in B_X$  and  $m, n \geq N$ . Taking  $m \rightarrow \infty$ , we have  $\|Tx - T_n x\| \leq \varepsilon$

for all  $n \geq N$  and  $x \in B_X$ . Therefore, we have  $\|T - T_n\| \leq \varepsilon$  for all  $n \geq N$ . From this, we see that  $T - T_N \in B(X, Y)$  and thus,  $T = T_N + (T - T_N) \in B(X, Y)$  and  $\|T - T_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $\lim_n T_n = T$  exists in  $B(X, Y)$ .  $\square$

#### 4. DUAL SPACES

By Proposition 3.12, we have the following assertion at once.

**Proposition 4.1.** *Let  $X$  be a normed space. Put  $X^* = B(X, \mathbb{K})$ . Then  $X^*$  is a Banach space and is called the dual space of  $X$ .*

**Example 4.2.** *Let  $X = \mathbb{K}^N$ . Consider the usual Euclidean norm on  $X$ , that is,  $\|(x_1, \dots, x_N)\| := \sqrt{|x_1|^2 + \dots + |x_N|^2}$ . Define  $\theta : \mathbb{K}^N \rightarrow (\mathbb{K}^N)^*$  by  $\theta x(y) = x_1 y_1 + \dots + x_N y_N$  for  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N) \in \mathbb{K}^N$ . Notice that  $\theta x(y) = \langle x, y \rangle$ , the usual inner product on  $\mathbb{K}^N$ . Then by the Cauchy-Schwarz inequality, it is easy to see that  $\theta$  is an isometric isomorphism. Therefore, we have  $\mathbb{K}^N = (\mathbb{K}^N)^*$ .*

**Example 4.3.** *Define a map  $T : \ell^1 \rightarrow c_0^*$  by*

$$(Tx)(\eta) = \sum_{i=1}^{\infty} x(i)\eta(i)$$

for  $x \in \ell^1$  and  $\eta \in c_0$ .

Then  $T$  is isometric isomorphism and hence,  $c_0^* = \ell^1$ .

*Proof.* The proof is divided into the following steps.

**Step 1.**  $Tx \in c_0^*$  for all  $x \in \ell^1$ .

In fact, let  $\eta \in c_0$ . Then

$$|Tx(\eta)| \leq \left| \sum_{i=1}^{\infty} x(i)\eta(i) \right| \leq \sum_{i=1}^{\infty} |x(i)||\eta(i)| \leq \|x\|_1 \|\eta\|_{\infty}.$$

So, *Step 1* follows.

**Step 2.**  $T$  is an isometry.

Notice that by *Step 1*, we have  $\|Tx\| \leq \|x\|_1$  for all  $x \in \ell^1$ . It needs to show that  $\|Tx\| \geq \|x\|_1$  for all  $x \in \ell^1$ . Fix  $x \in \ell^1$ . Now for each  $k = 1, 2, \dots$ , consider the polar form  $x(k) = |x(k)|e^{i\theta_k}$ . Notice that  $\eta_n := (e^{-i\theta_1}, \dots, e^{-i\theta_n}, 0, 0, \dots) \in c_0$  for all  $n = 1, 2, \dots$ . Then we have

$$\sum_{k=1}^n |x(k)| = \sum_{k=1}^n x(k)\eta_n(k) = Tx(\eta_n) = |Tx(\eta_n)| \leq \|Tx\|$$

for all  $n = 1, 2, \dots$ . So, we have  $\|x\|_1 \leq \|Tx\|$ .

**Step 3.**  $T$  is a surjection.

Let  $\phi \in c_0^*$  and let  $e_k \in c_0$  be given by  $e_k(j) = 1$  if  $j = k$ , otherwise, is equal to 0. Put  $x(k) := \phi(e_k)$  for  $k = 1, 2, \dots$  and consider the polar form  $x(k) = |x(k)|e^{i\theta_k}$  as above. Then we have

$$\sum_{k=1}^n |x(k)| = \phi\left(\sum_{k=1}^n e^{-i\theta_k} e_k\right) \leq \|\phi\| \left\| \sum_{k=1}^n e^{-i\theta_k} e_k \right\|_{\infty} = \|\phi\|$$

for all  $n = 1, 2, \dots$ . Therefore,  $x \in \ell^1$ .

Finally, we need to show that  $Tx = \phi$  and thus,  $T$  is surjective. In fact, if  $\eta = \sum_{k=1}^{\infty} \eta(k)e_k \in c_0$ ,

then we have

$$\phi(\eta) = \sum_{k=1}^{\infty} \eta(k)\phi(e_k) = \sum_{k=1}^{\infty} \eta(k)x_k = Tx(\eta).$$

So, the proof is finished by the *Steps* 1-3 above.  $\square$

**Example 4.4.** We have the other important examples of the dual spaces.

- (i)  $(\ell^1)^* = \ell^\infty$ .
- (ii) For  $1 < p < \infty$ ,  $(\ell^p)^* = \ell^q$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .
- (iii) For a locally compact Hausdorff space  $X$ ,  $C_0(X)^* = M(X)$ , where  $M(X)$  denotes the space of all regular Borel measures on  $X$ .

Parts (i) and (ii) can be obtained by the similar argument as in Example 4.3 (see also in [4, Chapter 8]). Part (iii) is known as the *Riesz representation Theorem* which is referred to [4, Section 21.5] for the details.

## 5. HAHN-BANACH THEOREM

All spaces  $X, Y, Z, \dots$  are normed spaces over the field  $\mathbb{K}$  throughout this section.

**Lemma 5.1.** *Let  $Y$  be a subspace of  $X$  and  $v \in X \setminus Y$ . Let  $Z = Y \oplus \mathbb{K}v$  be the linear span of  $Y$  and  $v$  in  $X$ . If  $f \in Y^*$ , then there is an extension  $F \in Z^*$  of  $f$  such that  $\|F\| = \|f\|$ .*

*Proof.* We may assume that  $\|f\| = 1$  by considering the normalization  $f/\|f\|$  if  $f \neq 0$ .

*Case  $\mathbb{K} = \mathbb{R}$ :*

We first note that since  $\|f\| = 1$ , we have  $|f(x) - f(y)| \leq \|(x + v) - (y + v)\|$  for all  $x, y \in Y$ . This implies that  $-f(x) - \|x + v\| \leq -f(y) + \|y + v\|$  for all  $x, y \in Y$ . Now let  $\gamma = \sup\{-f(x) - \|x + v\| : x \in X\}$ . This implies that  $\gamma$  exists and

$$(5.1) \quad -f(y) - \|y + v\| \leq \gamma \leq -f(y) + \|y + v\|$$

for all  $y \in Y$ . We define  $F : Z \rightarrow \mathbb{R}$  by  $F(y + \alpha v) := f(y) + \alpha\gamma$ . It is clear that  $F|_Y = f$ . For showing  $F \in Z^*$  with  $\|F\| = 1$ , since  $F|_Y = f$  on  $Y$  and  $\|f\| = 1$ , it needs to show  $|F(y + \alpha v)| \leq \|y + \alpha v\|$  for all  $y \in Y$  and  $\alpha \in \mathbb{R}$ .

In fact, for  $y \in Y$  and  $\alpha > 0$ , then by inequality 5.1, we have

$$(5.2) \quad |F(y + \alpha v)| = |f(y) + \alpha\gamma| \leq \|y + \alpha v\|.$$

Since  $y$  and  $\alpha$  are arbitrary in inequality 5.2, we see that  $|F(y + \alpha v)| \leq \|y + \alpha v\|$  for all  $y \in Y$  and  $\alpha \in \mathbb{R}$ . Therefore the result holds when  $\mathbb{K} = \mathbb{R}$ .

Now for the complex case, let  $h = \operatorname{Re} f$  and  $g = \operatorname{Im} f$ . Then  $f = h + ig$  and  $f, g$  both are real linear with  $\|h\| \leq 1$ . Note that since  $f(iy) = if(y)$  for all  $y \in Y$ , we have  $g(y) = -h(iy)$  for all  $y \in Y$ . This gives  $f(\cdot) = h(\cdot) - ih(i\cdot)$  on  $Y$ . Then by the real case above, there is a real linear extension  $H$  on  $Z := Y \oplus \mathbb{R}v \oplus i\mathbb{R}v$  of  $h$  such that  $\|H\| = \|h\|$ . Now define  $F : Z \rightarrow \mathbb{C}$  by  $F(\cdot) := H(\cdot) - iH(i\cdot)$ . Then  $F \in Z^*$  and  $F|_Y = f$ . Thus it remains to show that  $\|F\| = \|f\| = 1$ . It needs to show that  $|F(z)| \leq \|z\|$  for all  $z \in Z$ . Note for  $z \in Z$ , consider the polar form  $F(z) = re^{i\theta}$ . Then  $F(e^{-i\theta}z) = r \in \mathbb{R}$  and thus  $F(e^{-i\theta}z) = H(e^{-i\theta}z)$ . This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\| \|e^{-i\theta}z\| \leq \|z\|.$$

The proof is finished.  $\square$

**Remark 5.2.** Before completing the proof of the Hahn-Banach Theorem, Let us first recall one of super important results in mathematics, called *Zorn's Lemma*, a very humble name. Every mathematics student should know it.



**Zorn's Lemma:** Let  $\mathcal{X}$  be a non-empty set with a partially order " $\leq$ ". Assume that every totally order subset  $\mathcal{C}$  of  $\mathcal{X}$  has an upper bound, i.e. there is an element  $\mathfrak{z} \in \mathcal{X}$  such that  $c \leq \mathfrak{z}$  for all  $c \in \mathcal{C}$ . Then  $\mathcal{X}$  must contain a maximal element  $\mathfrak{m}$ , that is, if  $\mathfrak{m} \leq x$  for some  $x \in \mathcal{X}$ , then  $\mathfrak{m} = x$ .

The following is the typical argument of applying the Zorn's Lemma.

**Theorem 5.3. Hahn-Banach Theorem :** *Let  $X$  be a normed space and let  $Y$  be a subspace of  $X$ . If  $f \in Y^*$ , then there exists a linear extension  $F \in X^*$  of  $f$  such that  $\|F\| = \|f\|$ .*

*Proof.* Let  $\mathcal{X}$  be the collection of the pairs  $(Y_1, f_1)$ , where  $Y \subseteq Y_1$  is a subspace of  $X$  and  $f_1 \in Y_1^*$  such that  $f_1|_Y = f$  and  $\|f_1\|_{Y_1^*} = \|f\|_{Y^*}$ . Define a partial order  $\leq$  on  $\mathcal{X}$  by  $(Y_1, f_1) \leq (Y_2, f_2)$  if  $Y_1 \subseteq Y_2$  and  $f_2|_{Y_1} = f_1$ . Then by the Zorn's lemma, there is a maximal element  $(\tilde{Y}, \tilde{F})$  in  $\mathcal{X}$ . The maximality of  $(\tilde{Y}, \tilde{F})$  and Lemma 5.1 will give  $\tilde{Y} = X$ . The proof is finished.  $\square$

**Proposition 5.4.** *Let  $X$  be a normed space and  $x_0 \in X$ . Then there is  $f \in X^*$  with  $\|f\| = 1$  such that  $f(x_0) = \|x_0\|$ . Consequently, we have*

$$\|x_0\| = \sup\{|g(x)| : g \in B_{X^*}\}.$$

*Also, if  $x, y \in X$  with  $x \neq y$ , then there exists  $f \in X^*$  such that  $f(x) \neq f(y)$ .*

*Proof.* Let  $Y = \mathbb{K}x_0$ . Define  $f_0 : Y \rightarrow \mathbb{K}$  by  $f_0(\alpha x_0) := \alpha\|x_0\|$  for  $\alpha \in \mathbb{K}$ . Then  $f_0 \in Y^*$  with  $\|f_0\| = \|x_0\|$ . So, the result follows from the Hahn-Banach Theorem at once.  $\square$

**Remark 5.5.** Proposition 5.4 tells us that the dual space  $X^*$  of  $X$  must be non-zero. Indeed, the dual space  $X^*$  is very "Large" so that it can separate any pair of distinct points in  $X$ . Furthermore, for any normed space  $Y$  and any pair of points  $x_1, x_2 \in X$  with  $x_1 \neq x_2$ , we can find an element  $T \in B(X, Y)$  such that  $Tx_1 \neq Tx_2$ . In fact, fix a non-zero element  $y \in Y$ . Then by Proposition 5.4, there is  $f \in X^*$  such that  $f(x_1) \neq f(x_2)$ . So, if we define  $Tx = f(x)y$ , then  $T \in B(X, Y)$  as desired.

**Proposition 5.6.** *With the notation as above, if  $M$  is closed subspace and  $v \in X \setminus M$ , then there is  $f \in X^*$  such that  $f(M) \equiv 0$  and  $f(v) \neq 0$ .*

*Proof.* Since  $M$  is a closed subspace of  $X$ , we can consider the quotient space  $X/M$ . Let  $\pi : X \rightarrow X/M$  be the natural projection. Notice that  $\bar{v} := \pi(v) \neq 0 \in X/M$  because  $v \in X \setminus M$ . Then by Corollary 5.4, there is a non-zero element  $\bar{f} \in (X/M)^*$  such that  $\bar{f}(\bar{v}) \neq 0$ . So, the linear functional  $f := \bar{f} \circ \pi \in X^*$  is as desired.  $\square$

**Proposition 5.7.** *Using the notation as above, if  $X^*$  is separable, then  $X$  is separable.*

*Proof.* Let  $F := \{f_1, f_2, \dots\}$  be a dense subset of  $X^*$ . Then there is a sequence  $(x_n)$  in  $X$  with  $\|x_n\| = 1$  and  $|f_n(x_n)| \geq 1/2\|f_n\|$  for all  $n$ . Now let  $M$  be the closed linear span of  $x_n$ 's. Then  $M$  is a separable closed subspace of  $X$ . We are going to show that  $M = X$ .

Suppose not. Proposition 5.6 will give us a non-zero element  $f \in X^*$  such that  $f(M) \equiv 0$ . Since  $\{f_1, f_2, \dots\}$  is dense in  $X^*$ , we have  $B(f, r) \cap F \neq \emptyset$  for all  $r > 0$ . Therefore, if  $B(f, r) \cap F \neq \emptyset$  is finite for some  $r > 0$ , then  $f = f_m$  for some  $f_m \in F$ . This implies that  $\|f\| = \|f_m\| \leq 2|f_m(x_m)| = 2|f(x_m)| = 0$  and thus,  $f = 0$  which contradicts to  $f \neq 0$ .

So,  $B(f, r) \cap F$  is infinite for all  $r > 0$ . In this case, there is a subsequence  $(f_{n_k})$  such that  $\|f_{n_k} - f\| \rightarrow 0$ . This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \rightarrow 0$$

because  $f(M) \equiv 0$ . So  $\|f_{n_k}\| \rightarrow 0$  and hence  $f = 0$ . It leads to a contradiction again. Thus, we can conclude that  $M = X$  as desired.  $\square$

**Remark 5.8.** The converse of Proposition 5.7 does not hold. For example, consider  $X = \ell^1$ . Then  $\ell^1$  is separable but the dual space  $(\ell^1)^* = \ell^\infty$  is not.

**Proposition 5.9.** *Let  $X$  and  $Y$  be normed spaces. For each element  $T \in B(X, Y)$ , define a linear operator  $T^* : Y^* \rightarrow X^*$  by*

$$T^*y^*(x) := y^*(Tx)$$

for  $y^* \in Y^*$  and  $x \in X$ . Then  $T^* \in B(Y^*, X^*)$  and  $\|T^*\| = \|T\|$ . In this case,  $T^*$  is called the adjoint operator of  $T$ .

*Proof.* We first claim that  $\|T^*\| \leq \|T\|$  and hence,  $\|T^*\|$  is bounded.

In fact, for any  $y^* \in Y^*$  and  $x \in X$ , we have  $|T^*y^*(x)| = |y^*(Tx)| \leq \|y^*\| \|T\| \|x\|$ . So,  $\|T^*y^*\| \leq \|T\| \|y^*\|$  for all  $y^* \in Y^*$ . Thus,  $\|T^*\| \leq \|T\|$ .

It remains to show  $\|T\| \leq \|T^*\|$ . Let  $x \in B_X$ . Then by Proposition 5.4, there is  $y^* \in S_{Y^*}$  such that  $\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \leq \|T^*y^*\| \leq \|T^*\|$ . This implies that  $\|T\| \leq \|T^*\|$ .  $\square$

**Example 5.10.** Let  $X$  and  $Y$  be the finite dimensional normed spaces. Let  $(e_i)_{i=1}^n$  and  $(f_j)_{j=1}^m$  be the bases for  $X$  and  $Y$  respectively. Let  $\theta_X : X \rightarrow X^*$  and  $\theta_Y : Y \rightarrow Y^*$  be the identifications as in Example 4.2. Let  $e_i^* := \theta_X e_i \in X^*$  and  $f_j^* := \theta_Y f_j \in Y^*$ . Then  $e_i^*(e_l) = \delta_{il}$  and  $f_j^*(f_l) = \delta_{jl}$ , where,  $\delta_{il} = 1$  if  $i = l$ ; otherwise is 0.

Now if  $T \in B(X, Y)$  and  $(a_{ij})_{m \times n}$  is the representative matrix of  $T$  corresponding to the bases  $(e_i)_{i=1}^n$  and  $(f_j)_{j=1}^m$  respectively, then  $a_{kl} = f_k^*(Te_l) = T^*f_k^*(e_l)$ . Therefore, if  $(a'_{lk})_{n \times m}$  is the representative matrix of  $T^*$  corresponding to the bases  $(f_j^*)$  and  $(e_i^*)$ , then  $a_{kl} = a'_{lk}$ . Hence the transpose  $(a_{kl})^t$  is the representative matrix of  $T^*$ .

**Proposition 5.11.** *Let  $Y$  be a closed subspace of a normed space  $X$ . Let  $i : Y \rightarrow X$  be the natural inclusion and  $\pi : X \rightarrow X/Y$  the natural projection. Then*

(i) *the adjoint operator  $i^{**} : Y^{**} \rightarrow X^{**}$  is an isometry.*

(ii) *the adjoint operator  $\pi^* : (X/Y)^* \rightarrow X^*$  is an isometry.*

Consequently,  $Y^{**}$  and  $(X/Y)^*$  can be viewed as the closed subspaces of  $X^{**}$  and  $X^*$  respectively.

*Proof.* For Part (i), we first notice that for any  $x^* \in X^*$ , the image  $i^*x^*$  in  $Y^*$  is just the restriction of  $x^*$  on  $Y$ , write  $x^*|_Y$ . Now let  $\phi \in Y^{**}$ . Then for any  $x^* \in X^*$ , we have

$$|i^{**}\phi(x^*)| = |\phi(i^*x^*)| = |\phi(x^*|_Y)| \leq \|\phi\| \|x^*|_Y\|_{Y^*} \leq \|\phi\| \|x^*\|_{X^*}.$$

So,  $\|i^{**}\phi\| \leq \|\phi\|$ . It remains to show the inverse inequality. Now for each  $y^* \in Y^*$ , the Hahn-Banach Theorem gives an element  $x^* \in X^*$  such that  $\|x^*\|_{X^*} = \|y^*\|_{Y^*}$  and  $x^*|_Y = y^*$  and hence,  $i^*x^* = y^*$ . Then we have

$$|\phi(y^*)| = |\phi(x^*|_Y)| = |\phi(i^*x^*)| = |(i^{**} \circ \phi)(x^*)| \leq \|i^{**}\phi\| \|x^*\|_{X^*} = \|i^{**}\phi\| \|y^*\|_{Y^*}$$

for all  $y^* \in Y^*$ . Therefore, we have  $\|i^{**}\phi\| = \|\phi\|$ .

For Part (ii), let  $\psi \in (X/Y)^*$ . Notice that since  $\|\pi^*\| = \|\pi\| \leq 1$ , we have  $\|\pi^*\psi\| \leq \|\psi\|$ . On the other hand, for each  $\bar{x} := \pi(x) \in X/Y$  with  $\|\bar{x}\| < 1$ , we can choose an element  $m \in Y$  such that  $\|x + m\| < 1$ . So, we have

$$|\psi(\bar{x})| = |\psi \circ \pi(x)| = |\psi \circ \pi(x + m)| \leq \|\psi \circ \pi\| = \|\pi^*(\psi)\|.$$

Thus we have  $\|\psi\| \leq \|\pi^*(\psi)\|$ . The proof is finished.  $\square$

**Remark 5.12.** By using Proposition 5.11, we can give an alternative proof of the Riesz's Lemma 2.4.

With the notation as in Proposition 5.11, if  $Y \subsetneq X$ , then we have  $\|\pi\| = \|\pi^*\| = 1$  because  $\pi^*$  is an isometry by Proposition 5.11(ii). Thus we have  $\|\pi\| = \sup\{\|\pi(x)\| : x \in X, \|x\| = 1\} = 1$ . So, for any  $0 < \theta < 1$ , we can find element  $z \in X$  with  $\|z\| = 1$  such that  $\theta < \|\pi(z)\| = \inf\{\|z+y\| : y \in Y\}$ . The Riesz's Lemma follows.

## 6. REFLEXIVE SPACES

**Proposition 6.1.** *For a normed space  $X$ , let  $Q : X \rightarrow X^{**}$  be the canonical map, that is,  $Qx(x^*) := x^*(x)$  for  $x^* \in X^*$  and  $x \in X$ . Then  $Q$  is an isometry.*

*Proof.* Note that for  $x \in X$  and  $x^* \in B_{X^*}$ , we have  $|Q(x)(x^*)| = |x^*(x)| \leq \|x\|$ . Then  $\|Q(x)\| \leq \|x\|$ .

It remains to show that  $\|x\| \leq \|Q(x)\|$  for all  $x \in X$ . In fact, for  $x \in X$ , there is  $x^* \in X^*$  with  $\|x^*\| = 1$  such that  $\|x\| = |x^*(x)| = |Q(x)(x^*)|$  by Proposition 5.4. Thus we have  $\|x\| \leq \|Q(x)\|$ . The proof is finished.  $\square$

**Remark 6.2.** Let  $T : X \rightarrow Y$  be a bounded linear operator and  $T^{**} : X^{**} \rightarrow Y^{**}$  the second dual operator induced by the adjoint operator of  $T$ . With notation as in Proposition 6.1 above, the following diagram always commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ Q_X \downarrow & & \downarrow Q_Y \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

**Definition 6.3.** *A normed space  $X$  is said to be reflexive if the canonical map  $Q : X \rightarrow X^{**}$  is surjective. (Notice that every reflexive space must be a Banach space.)*

**Example 6.4.** *We have the following examples.*

- (i) : Every finite dimensional normed space  $X$  is reflexive.
- (ii) :  $\ell^p$  is reflexive for  $1 < p < \infty$ .
- (iii) :  $c_0$  and  $\ell^1$  are not reflexive.

*Proof.* For Part (i), if  $\dim X < \infty$ , then  $\dim X = \dim X^{**}$ . Hence, the canonical map  $Q : X \rightarrow X^{**}$  must be surjective.

Part (ii) follows from  $(\ell^p)^* = \ell^q$  for  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

For Part (iii), notice that  $c_0^{**} = (\ell^1)^* = \ell^\infty$ . Since  $\ell^\infty$  is non-separable but  $c_0$  is separable. So, the canonical map  $Q$  from  $c_0$  to  $c_0^{**} = \ell^\infty$  must not be surjective.

For the case of  $\ell^1$ , we have  $(\ell^1)^{**} = (\ell^\infty)^*$ . Since  $\ell^\infty$  is non-separable, the dual space  $(\ell^\infty)^*$  is non-separable by Proposition 5.7. So,  $\ell^1 \neq (\ell^1)^{**}$ .  $\square$

**Proposition 6.5.** *Every closed subspace of a reflexive space is reflexive.*

*Proof.* Let  $Y$  be a closed subspace of a reflexive space  $X$ . Let  $Q_Y : Y \rightarrow Y^{**}$  and  $Q_X : X \rightarrow X^{**}$  be the canonical maps as before. Let  $y_0^{**} \in Y^{**}$ . We define an element  $\phi \in X^{**}$  by  $\phi(x^*) := y_0^{**}(x^*|_Y)$  for  $x^* \in X^*$ . Since  $X$  is reflexive, there is  $x_0 \in X$  such that  $Q_X x_0 = \phi$ . Suppose  $x_0 \notin Y$ . Then by Proposition 5.6, there is  $x_0^* \in X^*$  such that  $x_0^*(x_0) \neq 0$  but  $x_0^*(Y) \equiv 0$ . Note that we have  $x_0^*(x_0) = Q_X x_0(x_0^*) = \phi(x_0^*) = y_0^{**}(x_0^*|_Y) = 0$ . It leads to a contradiction. So,  $x_0 \in Y$ . The proof is finished if we have  $Q_Y(x_0) = y_0^{**}$ .

In fact, for each  $y^* \in Y^*$ , then by the Hahn-Banach Theorem,  $y^*$  has a continuous extension  $x^*$  in  $X^*$ . Then we have

$$Q_Y(x_0)(y^*) = y^*(x_0) = x^*(x_0) = Q_X(x_0)(x^*) = \phi(x^*) = y_0^{**}(x^*|_Y) = y_0^{**}(y^*).$$

□

**Example 6.6.** *By using Proposition 6.5, we immediately see that the space  $\ell^\infty$  is not reflexive because it contains a non-reflexive closed subspace  $c_0$ .*

**Proposition 6.7.** *Let  $X$  be a Banach space. Then we have the following assertions.*

- (i)  $X$  is reflexive if and only if the dual space  $X^*$  is reflexive.
- (ii) If  $X$  is reflexive, then so is every quotient of  $X$ .

*Proof.* For Part (i), suppose that  $X$  is reflexive first. Let  $\tilde{z} \in X^{***}$ . Then the restriction  $z := \tilde{z}|_X \in X^*$ . Then one can directly check that  $Qz = z$  on  $X^{**}$  since  $X^{**} = X$ .

For the converse, assume that  $X^*$  is reflexive but  $X$  is not. So,  $X$  is a proper closed subspace of  $X^{**}$ . Then by using the Hahn-Banach Theorem, we can find a non-zero element  $\phi \in X^{***}$  such that  $\phi(X) \equiv 0$ . However, since  $X^{***}$  is reflexive, we have  $\phi \in X^*$  and hence,  $\phi = 0$  which leads to a contradiction.

For Part (ii), we assume that  $X$  is reflexive. Let  $M$  be a closed subspace of  $X$  and  $\pi : X \rightarrow X/M$  the natural projection. Notice that the adjoint operator  $\pi^* : (X/M)^* \rightarrow X^*$  is an isometry (**Check !**). So,  $(X/M)^*$  can be viewed as a closed subspace of  $X^*$ . So, by Part (i) and Proposition 6.5, we see that  $(X/M)^*$  is reflexive. Then  $X/M$  is reflexive by using Part (i) again.

The proof is complete. □

**Lemma 6.8.** *Let  $M$  be a closed subspace of a normed space  $X$ . Let  $r : X^* \rightarrow M^*$  be the restriction map, that is  $x^* \in X^* \mapsto x^*|_M \in M^*$ . Put  $M^\perp := \ker r := \{x^* \in X^* : x^*(M) \equiv 0\}$ . Then the canonical linear isomorphism  $\tilde{r} : X^*/M^\perp \rightarrow M^*$  induced by  $r$  is an isometric isomorphism.*

*Proof.* We first note that  $r$  is surjective by using the Hahn-Banach Theorem. It needs to show that  $\tilde{r}$  is an isometry. Notice that  $\tilde{r}(x^* + M^\perp) = x^*|_M$  for all  $x^* \in X^*$ . Now for any  $x^* \in X^*$ , we have  $\|x^* + y^*\|_{X^*} \geq \|x^* + y^*\|_{M^*} = \|x^*|_M\|_{M^*}$  for all  $y^* \in M^\perp$ . So we have  $\|\tilde{r}(x^* + M^\perp)\| = \|x^*|_M\|_{M^*} \leq \|x^* + M^\perp\|$ . It remains to show the reverse inequality.

Now for any  $x^* \in X^*$ , then by the Hahn-Banach Theorem again, there is  $z^* \in X^*$  such that  $z^*|_M = x^*|_M$  and  $\|z^*\| = \|x^*|_M\|_{M^*}$ . Then  $x^* - z^* \in M^\perp$  and hence, we have  $x^* + M^\perp = z^* + M^\perp$ . This implies that

$$\|x^* + M^\perp\| = \|z^* + M^\perp\| \leq \|z^*\| = \|x^*|_M\|_{M^*} = \|\tilde{r}(x^* + M^\perp)\|.$$

The proof is complete. □

**Proposition 6.9. (Three space property):** *Let  $M$  be a closed subspace of a normed space  $X$ . If  $M$  and the quotient space  $X/M$  both are reflexive, then so is  $X$ .*

*Proof.* Let  $\pi : X \rightarrow X/M$  be the natural projection. Let  $\psi \in X^{**}$ . We going to show that  $\psi \in \text{im}(Q_X)$ . Since  $\pi^{**}(\psi) \in (X/M)^{**}$ , there exists  $x_0 \in X$  such that  $\pi^{**}(\psi) = Q_{X/M}(x_0 + M)$  because  $X/M$  is reflexive. So we have

$$\pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*)$$

for all  $\bar{x}^* \in (X/M)^*$ . This implies that

$$\psi(\bar{x}^* \circ \pi) = \psi(\pi^* \bar{x}^*) = \pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*) = \bar{x}^*(x_0 + M) = Q_X x_0(\bar{x}^* \circ \pi)$$

for all  $\bar{x}^* \in (X/M)^*$ . Therefore, we have

$$\psi = Q_X x_0 \quad \text{on} \quad M^\perp.$$

So, we have  $\psi - Q_X(x_0) \in X^*/M^\perp$ . Let  $f : M^* \rightarrow X^*/M^\perp$  be the inverse of the isometric isomorphism  $\tilde{r}$  which is defined as in Lemma 6.8. Then the composite  $(\psi - Q_X x_0) \circ f : M^* \rightarrow X^*/M^\perp \rightarrow \mathbb{K}$  lies in  $M^{**}$ . Then by the reflexivity of  $M$ , there is an element  $m_0 \in M$  such that

$$(\psi - Q_X x_0) \circ f = Q_M(m_0) \in M^{**}.$$

On the other hand, notice that for each  $x^* \in X^*$ , we can find an element  $m^* \in M^*$  such that  $f(m^*)x^* + M|_{\text{bot}} \in X^*/M^\perp$  because  $f$  is surjective, moreover, by the construction of  $\tilde{r}$  in Lemma 6.8, we see that  $x^*|_M = m^*$ . This gives

$$\psi(x^*) - x^*(x_0) = (\psi - Q_X x_0)(m^*) \circ f = Q_M(m_0)(m^*) = m^*(m_0) = x^*(m_0).$$

Thus, we have  $\psi(x^*) = x^*(x_0 + m_0)$  for all  $x^* \in X^*$ . From this we have  $\psi = Q_X(x_0 + m_0) \in \text{im}(Q_X)$  as desired. The proof is complete.  $\square$

## 7. WEAKLY CONVERGENT AND WEAK\* CONVERGENT

**Definition 7.1.** Let  $X$  be a normed space. A sequence  $(x_n)$  is said to be weakly convergent if there is  $x \in X$  such that  $f(x_n) \rightarrow f(x)$  for all  $f \in X^*$ . In this case,  $x$  is called a weak limit of  $(x_n)$ .

**Proposition 7.2.** A weak limit of a sequence is unique if it exists. In this case, if  $(x_n)$  weakly converges to  $x$ , write  $x = w\text{-}\lim_n x_n$  or  $x_n \xrightarrow{w} x$ .

*Proof.* The uniqueness follows from the Hahn-Banach Theorem immediately.  $\square$

**Remark 7.3.** It is clear that if a sequence  $(x_n)$  converges to  $x \in X$  in norm, then  $x_n \xrightarrow{w} x$ . However, the weakly convergence of a sequence does not imply the norm convergence. For example, consider  $X = c_0$  and  $(e_n)$ . Then  $f(e_n) \rightarrow 0$  for all  $f \in c_0^* = \ell^1$  but  $(e_n)$  is not convergent in  $c_0$ .

**Proposition 7.4.** Suppose that  $X$  is finite dimensional. A sequence  $(x_n)$  in  $X$  is norm convergent if and only if it is weakly convergent.

*Proof.* Suppose that  $(x_n)$  weakly converges to  $x$ . Let  $\mathcal{B} := \{e_1, \dots, e_N\}$  be a base for  $X$  and let  $f_k$  be the  $k$ -th coordinate functional corresponding to the base  $\mathcal{B}$ , that is  $v = \sum_{k=1}^N f_k(v)e_k$  for all  $v \in X$ . Since  $\dim X < \infty$ , we have  $f_k \in X^*$  for all  $k = 1, \dots, N$ . Therefore, we have  $\lim_n f_k(x_n) = f_k(x)$  for all  $k = 1, \dots, N$ . So, we have  $\|x_n - x\| \rightarrow 0$ .  $\square$

**Definition 7.5.** Let  $X$  be a normed space. A sequence  $(f_n)$  in  $X^*$  is said to be weak\* convergent if there is  $f \in X^*$  such that  $\lim_n f_n(x) = f(x)$  for all  $x \in X$ , that is  $f_n$  point-wise converges to  $f$ . In this case,  $f$  is called the weak\* limit of  $(f_n)$ . Write  $f = w^*\text{-}\lim_n f_n$  or  $f_n \xrightarrow{w^*} f$ .

**Remark 7.6.** In the dual space  $X^*$  of a normed space  $X$ , we always have the following implications:

$$\text{“Norm Convergent”} \implies \text{“Weakly Convergent”} \implies \text{“Weak* Convergent”}.$$

However, the converse of each implication does not hold.

**Example 7.7.** Remark 7.3 has shown that the  $w$ -convergence does not imply  $\|\cdot\|$ -convergence.

We now claim that the  $w^*$ -convergence also Does Not imply the  $w$ -convergence.

Consider  $X = c_0$ . Then  $c_0^* = \ell^1$  and  $c_0^{**} = (\ell^1)^* = \ell^\infty$ . Let  $e_n^* = (0, \dots, 0, 1, 0, \dots) \in \ell^1 = c_0^*$ , where the  $n$ -th coordinate is 1. Then  $e_n^* \xrightarrow{w^*} 0$  but  $e_n^* \not\xrightarrow{w} 0$  weakly because  $e_n^{**}(e_n^*) \equiv 1$  for all  $n$ , where  $e_n^{**} := (1, 1, \dots) \in \ell^\infty = c_0^{**}$ . Hence the  $w^*$ -convergence does not imply the  $w$ -convergence.

**Proposition 7.8.** *Let  $(f_n)$  be a sequence in  $X^*$ . Suppose that  $X$  is reflexive. Then  $f_n \xrightarrow{w} f$  if and only if  $f_n \xrightarrow{w^*} f$ .*

*In particular, if  $\dim X < \infty$ , then the followings are equivalent:*

- (i) :  $f_n \xrightarrow{\|\cdot\|} f$ ;
- (ii) :  $f_n \xrightarrow{w} f$ ;
- (iii) :  $f_n \xrightarrow{w^*} f$ .

**Theorem 7.9. (Banach)** : *Let  $X$  be a separable normed space. If  $(f_n)$  is a bounded sequence in  $X^*$ , then it has a  $w^*$ -convergent subsequence.*

*Proof.* Let  $D := \{x_1, x_2, \dots\}$  be a countable dense subset of  $X$ . Note that since  $(f_n)_{n=1}^\infty$  is bounded,  $(f_n(x_1))$  is a bounded sequence in  $\mathbb{K}$ . Then  $(f_n(x_1))$  has a convergent subsequence, say  $(f_{1,k}(x_1))_{k=1}^\infty$  in  $\mathbb{K}$ . Let  $c_1 := \lim_k f_{1,k}(x_1)$ . Now consider the bounded sequence  $(f_{1,k}(x_2))$ . Then there is convergent subsequence, say  $(f_{2,k}(x_2))$ , of  $(f_{1,k}(x_2))$ . Put  $c_2 := \lim_k f_{2,k}(x_2)$ . Notice that we still have  $c_1 = \lim_k f_{2,k}(x_1)$ . To repeat the same step, if we define  $(m, k) \leq (m', k')$  if  $m < m'$ ; or  $m = m'$  with  $k \leq k'$ , we can find a sequence  $(f_{m,k})_{m,k}$  in  $X^*$  such that

- (i) :  $(f_{m+1,k})_{k=1}^\infty$  is a subsequence of  $(f_{m,k})_{k=1}^\infty$  for  $m = 0, 1, \dots$ , where  $f_{0,k} := f_k$ .
- (ii) :  $c_i = \lim_k f_{m,k}(x_i)$  exists for all  $1 \leq i \leq m$ .

Now put  $h_k := f_{k,k}$ . Then  $(h_k)$  is a subsequence of  $(f_n)$ . Notice that for each  $i$ , we have  $\lim_k h_k(x_i) = \lim_k f_{i,k}(x_i) = c_i$  by the construction (ii) above. Since  $(\|h_k\|)$  is bounded and  $D$  is dense in  $X$ , we have  $h(x) := \lim_k h_k(x)$  exists for all  $x \in X$  and  $h \in X^*$ . That is  $h = w^*\text{-}\lim_k h_k$ . The proof is finished.  $\square$

**Remark 7.10.** *Theorem 7.9 does not hold if the separability of  $X$  is removed.*

*For example, consider  $X = \ell^\infty$  and  $\delta_n$  the  $n$ -th coordinate functional on  $\ell^\infty$ . Then  $\delta_n \in (\ell^\infty)^*$  with  $\|\delta_n\|_{(\ell^\infty)^*} = 1$  for all  $n$ . Suppose that  $(\delta_n)$  has a  $w^*$ -convergent subsequence  $(\delta_{n_k})_{k=1}^\infty$ . Define  $x \in \ell^\infty$  by*

$$x(m) = \begin{cases} 0 & \text{if } m \neq n_k; \\ 1 & \text{if } m = n_{2k}; \\ -1 & \text{if } m = n_{2k+1}. \end{cases}$$

*Hence we have  $|\delta_{n_i}(x) - \delta_{n_{i+1}}(x)| = 2$  for all  $i = 1, 2, \dots$ . It leads to a contradiction. So  $(\delta_n)$  has no  $w^*$ -convergent subsequence.*

**Corollary 7.11.** *Let  $X$  be a separable space. In  $X^*$  assume that the set of all  $w^*$ -convergent sequences coincides with the set of all normed convergent sequences, that is a sequence  $(f_n)$  is  $w^*$ -convergent if and only if it is norm convergent. Then  $\dim X < \infty$ .*

*Proof.* It needs to show that the closed unit ball  $B_{X^*}$  in  $X^*$  is compact in norm. Let  $(f_n)$  be a sequence in  $B_{X^*}$ . By using Theorem 7.9,  $(f_n)$  has a  $w^*$ -convergent subsequence  $(f_{n_k})$ . Then by the assumption,  $(f_{n_k})$  is norm convergent. Note that if  $\lim_k f_{n_k} = f$  in norm, then  $f \in B_{X^*}$ . So  $B_{X^*}$  is compact and thus  $\dim X^* < \infty$ . So  $\dim X^{**} < \infty$  that gives  $\dim X$  is finite because  $X \subseteq X^{**}$ .  $\square$

**Corollary 7.12.** *Suppose that  $X$  is a separable. If  $X$  is reflexive space, then the closed unit ball  $B_X$  of  $X$  is sequentially weakly compact, i.e. it is equivalent to saying that any bounded sequence in  $X$  has a weakly convergent subsequence.*

*Proof.* Let  $Q : X \rightarrow X^{**}$  be the canonical map as before. Let  $(x_n)$  be a bounded sequence in  $X$ . Hence,  $(Qx_n)$  is a bounded sequence in  $X^{**}$ . We first notice that since  $X$  is reflexive and separable,  $X^*$  is also separable by Proposition 5.7. So, we can apply Theorem 7.9,  $(Qx_n)$  has a  $w^*$ -convergent subsequence  $(Qx_{n_k})$  in  $X^{**} = Q(X)$  and hence,  $(x_{n_k})$  is weakly convergent in  $X$ .  $\square$

## 8. OPEN MAPPING THEOREM

Let  $E$  and  $F$  be the metric spaces. Recall that a mapping  $f : E \rightarrow F$  is called an *open mapping* if  $f(U)$  is an open subset of  $F$  whenever  $U$  is an open subset of  $E$ .

It is clear that a continuous bijection is a homeomorphism if and only if it is an open map.

**Remark 8.1. Warning** *An open map need not be a closed map.*

*For example, let  $p : (x, y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$ . Then  $p$  is an open map but it is not a closed map. In fact, if we let  $A = \{(x, 1/x) : x \neq 0\}$ , then  $A$  is closed but  $p(A) = \mathbb{R} \setminus \{0\}$  is not closed.*

**Lemma 8.2.** *Let  $X$  and  $Y$  be normed spaces and  $T : X \rightarrow Y$  a linear map. Then  $T$  is open if and only if  $0$  is an interior point of  $T(U)$  where  $U$  is the open unit ball of  $X$ .*

*Proof.* The necessary condition is obvious.

For the converse, let  $W$  be a non-empty subset of  $X$  and  $a \in W$ . Put  $b = Ta$ . Since  $W$  is open, we choose  $r > 0$  such that  $B_X(a, r) \subseteq W$ . Notice that  $U = \frac{1}{r}(B_X(a, r) - a) \subseteq \frac{1}{r}(W - a)$ . So, we have  $T(U) \subseteq \frac{1}{r}(T(W) - b)$ . Then by the assumption, there is  $\delta > 0$  such that  $B_Y(0, \delta) \subseteq T(U) \subseteq \frac{1}{r}(T(W) - b)$ . This implies that  $b + rB_Y(0, \delta) \subseteq T(W)$  and so,  $T(a) = b$  is an interior point of  $T(W)$ .  $\square$

**Corollary 8.3.** *Let  $M$  be a closed subspace of a normed space  $X$ . Then the natural projection  $\pi : X \rightarrow X/M$  is an open map.*

*Proof.* Put  $U$  and  $V$  the open unit balls of  $X$  and  $X/M$  respectively. Using Lemma 8.2, the result is obtained by showing that  $V \subseteq \pi(U)$ . Note that if  $\bar{x} = \pi(x) \in V$ , then by the definition a quotient norm, we can find an element  $m \in M$  such that  $\|x + m\| < 1$ . Hence we have  $x + m \in U$  and  $\bar{x} = \pi(x + m) \in \pi(U)$ .  $\square$

**Lemma 8.4.** *Let  $T : X \rightarrow Y$  be a bounded linear surjection from a Banach space  $X$  onto a Banach space  $Y$ . Then  $0$  is an interior point of  $T(U)$ , where  $U$  is the open unit ball of  $X$ , that is,  $U := \{x \in X : \|x\| < 1\}$ .*

*Proof.* Set  $U(r) := \{x \in X : \|x\| < r\}$  for  $r > 0$  and so,  $U = U(1)$ .

**Claim 1 :**  $0$  is an interior point of  $\overline{T(U(1))}$ .

Note that since  $T$  is surjective,  $Y = \bigcup_{n=1}^{\infty} T(U(n))$ . Then by the second category theorem, there exists  $N$  such that  $\text{int } \overline{T(U(N))} \neq \emptyset$ . Let  $y'$  be an interior point of  $\overline{T(U(N))}$ . Then there is  $\eta > 0$  such that  $B_Y(y', \eta) \subseteq \overline{T(U(N))}$ . Since  $B_Y(y', \eta) \cap T(U(N)) \neq \emptyset$ , we may assume that  $y' \in T(U(N))$ . Let  $x' \in U(N)$  such that  $T(x') = y'$ . Then we have

$$0 \in B_Y(y', \eta) - y' \subseteq \overline{T(U(N))} - T(x') \subseteq \overline{T(U(2N))} = 2N\overline{T(U(1))}.$$

So we have  $0 \in \frac{1}{2N}(B_Y(y', \eta) - y') \subseteq \overline{T(U(1))}$ . Hence  $0$  is an interior point of  $\overline{T(U(1))}$ . So Claim 1 follows.

Therefore there is  $r > 0$  such that  $B_Y(0, r) \subseteq \overline{T(U(1))}$ . This implies that we have

$$(8.1) \quad B_Y(0, r/2^k) \subseteq \overline{T(U(1/2^k))}$$

for all  $k = 0, 1, 2, \dots$

**Claim 2 :**  $D := B_Y(0, r) \subseteq T(U(3))$ .

Let  $y \in D$ . By Eq 8.1, there is  $x_1 \in U(1)$  such that  $\|y - T(x_1)\| < r/2$ . Then by using Eq 8.1 again, there is  $x_2 \in U(1/2)$  such that  $\|y - T(x_1) - T(x_2)\| < r/2^2$ . To repeat the same steps, there exists a sequence  $(x_k)$  such that  $x_k \in U(1/2^{k-1})$  and

$$\|y - T(x_1) - T(x_2) - \dots - T(x_k)\| < r/2^k$$

for all  $k$ . On the other hand, since  $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 1/2^{k-1}$  and  $X$  is Banach,  $x := \sum_{k=1}^{\infty} x_k$  exists in  $X$  and  $\|x\| \leq 2$ . This implies that  $y = T(x)$  and  $\|x\| < 3$ .

Thus we the result follows.  $\square$

**Theorem 8.5. Open Mapping Theorem :** *Retains the notation as in Lemma 8.4. Then  $T$  is an open mapping.*

*Proof.* The proof is finished by using Lemmas 8.2 and 8.4 at once.  $\square$

**Proposition 8.6.** *Let  $T$  be a bounded linear isomorphism between Banach spaces  $X$  and  $Y$ . Then  $T^{-1}$  must be bounded.*

*Consequently, if  $\|\cdot\|$  and  $\|\cdot\|'$  both are complete norms on  $X$  such that  $\|\cdot\| \leq c\|\cdot\|'$  for some  $c > 0$ , then these two norms  $\|\cdot\|$  and  $\|\cdot\|'$  are equivalent.*

*Proof.* The first assertion follows from the Open Mapping Theorem at once.

Therefore, the last assertion can be obtained by considering the identity map  $I : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$  which is bounded by the assumption.  $\square$

**Corollary 8.7.** *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  a bounded linear operator. Then the image of  $T$  is closed in  $Y$  if and only if there is  $c > 0$  such that*

$$d(x, \ker T) \leq c\|Tx\|$$

for all  $x \in X$ .

*Proof.* Let  $Z$  be the image of  $T$ . Then the canonical map  $\tilde{T} : X/\ker T \rightarrow Z$  induced by  $T$  is a bounded linear isomorphism. Notice that  $\tilde{T}(\bar{x}) = Tx$  for all  $x \in X$ , where  $\bar{x} := x + \ker T \in X/\ker T$ . Now suppose that  $Z$  is closed. Then  $Z$  becomes a Banach space. Then the Open Mapping Theorem implies that the inverse of  $\tilde{T}$  is also bounded. So, there is  $c > 0$  such that  $d(x, \ker T) = \|\bar{x}\|_{X/\ker T} \leq c\|\tilde{T}(\bar{x})\| = c\|T(x)\|$  for all  $x \in X$ . So, the necessary condition follows.

For the converse, let  $(x_n)$  be a sequence in  $X$  such that  $\lim Tx_n = y \in Y$  exists and so,  $(Tx_n)$  is a Cauchy sequence in  $Y$ . Then by the assumption,  $(\bar{x}_n)$  is a Cauchy sequence in  $X/\ker T$ . Since  $X/\ker T$  is complete, we can find an element  $x \in X$  such that  $\lim \bar{x}_n = \bar{x}$  in  $X/\ker T$ . This gives  $y = \lim T(x_n) = \lim \tilde{T}(\bar{x}_n) = \tilde{T}(\bar{x}) = T(x)$ . So,  $y \in Z$ . The proof is finished.  $\square$

## 9. CLOSED GRAPH THEOREM

Let  $T : X \rightarrow Y$ . The *graph* of  $T$ , write  $\mathcal{G}(T)$  is defined by the set  $\{(x, y) \in X \times Y : y = T(x)\}$ . Now the direct sum  $X \oplus Y$  is endowed with the norm  $\|\cdot\|_{\infty}$ , that is  $\|x \oplus y\|_{\infty} := \max(\|x\|_X, \|y\|_Y)$ . We also write  $X \oplus_{\infty} Y$  when  $X \oplus Y$  is equipped with this norm.

We say that an operator  $T : X \rightarrow Y$  is said to be closed if its graph  $\mathcal{G}(T)$  is a closed subset of  $X \oplus_{\infty} Y$ , that is, if a sequence  $(x_n)$  of  $X$  satisfying the condition  $\|(x_n, Tx_n) - (x, y)\|_{\infty} \rightarrow 0$  for some  $x \in X$  and  $y \in Y$  implies  $T(x) = y$ .

**Theorem 9.1. Closed Graph Theorem :** *Let  $T : X \rightarrow Y$  be a linear operator from a Banach space  $X$  to a Banach  $Y$ . Then  $T$  is bounded if and only if  $T$  is closed.*

*Proof.* The part  $(\Rightarrow)$  is clear.

Assume that  $T$  is closed, that is, the graph  $\mathcal{G}(T)$  is  $\|\cdot\|_{\infty}$ -closed. Define  $\|\cdot\|_0 : X \rightarrow [0, \infty)$  by

$$\|x\|_0 = \|x\| + \|T(x)\|$$

for  $x \in X$ . Then  $\|\cdot\|_0$  is a norm on  $X$ . Let  $I : (X, \|\cdot\|_0) \rightarrow (X, \|\cdot\|)$  be the identity operator. It is clear that  $I$  is bounded since  $\|\cdot\| \leq \|\cdot\|_0$ .

**Claim:**  $(X, \|\cdot\|_0)$  is Banach. In fact, let  $(x_n)$  be a Cauchy sequence in  $(X, \|\cdot\|_0)$ . Then  $(x_n)$  and  $(T(x_n))$  both are Cauchy sequences in  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|_Y)$ . Since  $X$  and  $Y$  are Banach spaces,



there are  $x \in X$  and  $y \in Y$  such that  $\|x_n - x\|_X \rightarrow 0$  and  $\|T(x_n) - y\|_Y \rightarrow 0$ . Thus  $y = T(x)$  since the graph  $\mathcal{G}(T)$  is closed.

Then by Theorem 8.6, the norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent. So, there is  $c > 0$  such that  $\|T(\cdot)\| \leq \|\cdot\|_0 \leq c\|\cdot\|$  and hence,  $T$  is bounded since  $\|T(\cdot)\| \leq \|\cdot\|_0$ . The proof is finished.  $\square$

**Example 9.2.** Let  $D := \{\mathbf{c} = (c_n) \in \ell^2 : \sum_{n=1}^{\infty} n^2 |c_n|^2 < \infty\}$ . Define  $T : D \rightarrow \ell^2$  by  $T(\mathbf{c}) = (nc_n)$ . Then  $T$  is an unbounded closed operator.

*Proof.* Note that since  $\|Te_n\| = n$  for all  $n$ ,  $T$  is not bounded. Now we claim that  $T$  is closed. Let  $(\mathbf{x}_i)$  be a convergent sequence in  $D$  such that  $(T\mathbf{x}_i)$  is also convergent in  $\ell^2$ . Write  $\mathbf{x}_i = (x_{i,n})_{n=1}^{\infty}$  with  $\lim_i \mathbf{x}_i = \mathbf{x} := (x_n)$  in  $D$  and  $\lim_i T\mathbf{x}_i = \mathbf{y} := (y_n)$  in  $\ell^2$ . This implies that if we fix  $n_0$ , then  $\lim_i x_{i,n_0} = x_{n_0}$  and  $\lim_i n_0 x_{i,n_0} = y_{n_0}$ . This gives  $n_0 x_{n_0} = y_{n_0}$ . Thus  $T\mathbf{x} = \mathbf{y}$  and hence  $T$  is closed.  $\square$

**Example 9.3.** Let  $X := \{f \in C^b(0,1) \cap C^\infty(0,1) : f' \in C^b(0,1)\}$ . Define  $T : f \in X \mapsto f' \in C^b(0,1)$ . Suppose that  $X$  and  $C^b(0,1)$  both are equipped with the sup-norm. Then  $T$  is a closed unbounded operator.

*Proof.* Note that if a sequence  $f_n \rightarrow f$  in  $X$  and  $f'_n \rightarrow g$  in  $C^b(0,1)$ . Then  $f' = g$ . Hence  $T$  is closed. In fact, if we fix some  $0 < c < 1$ , then by the Fundamental Theorem of Calculus, we have

$$0 = \lim_n (f_n(x) - f(x)) = \lim_n \left( \int_c^x (f'_n(t) - f'(t)) dt \right) = \int_c^x (g(t) - f'(t)) dt$$

for all  $x \in (0,1)$ . This implies that we have  $\int_c^x g(t) dt = \int_c^x f'(t) dt$ . So  $g = f'$  on  $(0,1)$ . On the other hand, since  $\|Tx^n\|_\infty = n$  for all  $n \in \mathbb{N}$ . Thus  $T$  is unbounded as desired.  $\square$

## 10. UNIFORM BOUNDEDNESS THEOREM

**Theorem 10.1. Uniform Boundedness Theorem :** Let  $\{T_i : X \rightarrow Y : i \in I\}$  be a family of bounded linear operators from a Banach space  $X$  into a normed space  $Y$ . Suppose that for each  $x \in X$ , we have  $\sup_{i \in I} \|T_i(x)\| < \infty$ . Then  $\sup_{i \in I} \|T_i\| < \infty$ .

*Proof.* For each  $x \in X$ , define

$$\|x\|_0 := \max(\|x\|, \sup_{i \in I} \|T_i(x)\|).$$

Then  $\|\cdot\|_0$  is a norm on  $X$  and  $\|\cdot\| \leq \|\cdot\|_0$  on  $X$ . If  $(X, \|\cdot\|_0)$  is complete, then by the Open Mapping Theorem. This implies that  $\|\cdot\|$  is equivalent to  $\|\cdot\|_0$  and thus there is  $c > 0$  such that

$$\|T_j(x)\| \leq \sup_{i \in I} \|T_i(x)\| \leq \|x\|_0 \leq c\|x\|$$

for all  $x \in X$  and for all  $j \in I$ . So  $\|T_j\| \leq c$  for all  $j \in I$  is as desired.

Thus it remains to show that  $(X, \|\cdot\|_0)$  is complete. In fact, if  $(x_n)$  is a Cauchy sequence in  $(X, \|\cdot\|_0)$ , then it is also a Cauchy sequence with respect to the norm  $\|\cdot\|$  on  $X$ . Write  $x := \lim_n x_n$  with respect to the norm  $\|\cdot\|$ . Also for any  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that  $\|T_i(x_n - x_m)\| < \varepsilon$  for all  $m, n \geq N$  and for all  $i \in I$ . Now fixing  $i \in I$  and  $n \geq N$  and taking  $m \rightarrow \infty$ , we have  $\|T_i(x_n - x)\| \leq \varepsilon$  and thus  $\sup_{i \in I} \|T_i(x_n - x)\| \leq \varepsilon$  for all  $n \geq N$ . So we have  $\|x_n - x\|_0 \rightarrow 0$  and hence  $(X, \|\cdot\|_0)$  is complete. The proof is finished.  $\square$

**Remark 10.2.** Consider  $c_{00} := \{\mathbf{x} = (x_n) : \exists N, \forall n \geq N; x_n \equiv 0\}$  which is endowed with  $\|\cdot\|_\infty$ . Now for each  $k \in \mathbb{N}$ , if we define  $T_k \in c_{00}^*$  by  $T_k((x_n)) := kx_k$ , then  $\sup_k |T_k(\mathbf{x})| < \infty$  for each  $\mathbf{x} \in c_{00}$  but  $(\|T_k\|)$  is not bounded, in fact,  $\|T_k\| = k$ . Thus the assumption of the completeness of  $X$  in Theorem 10.1 is essential.

**Corollary 10.3.** *Let  $X$  and  $Y$  be as in Theorem 10.1. Let  $T_k : X \rightarrow Y$  be a sequence of bounded operators. Assume that  $\lim_k T_k(x)$  exists in  $Y$  for all  $x \in X$ . Then there is  $T \in B(X, Y)$  such that  $\lim_k \|(T - T_k)x\| = 0$  for all  $x \in X$ . Moreover, we have  $\|T\| \leq \liminf_k \|T_k\|$ .*

*Proof.* Notice that by the assumption, we can define a linear operator  $T$  from  $X$  to  $Y$  given by  $Tx := \lim_k T_k x$  for  $x \in X$ . It needs to show that  $T$  is bounded. In fact,  $(\|T_k\|)$  is bounded by the Uniform Boundedness Theorem since  $\lim_k T_k x$  exists for all  $x \in X$ . So for each  $x \in B_X$ , there is a positive integer  $K$  such that  $\|Tx\| \leq \|T_K x\| + 1 \leq (\sup_k \|T_k\|) + 1$ . Thus,  $T$  is bounded.

Finally, it remains to show the last assertion. In fact, notice that for any  $x \in B_X$  and  $\varepsilon > 0$ , there is  $N(x) \in \mathbb{N}$  such that  $\|Tx\| < \|T_k x\| + \varepsilon < \|T_k\| + \varepsilon$  for all  $k \geq N(x)$ . This gives  $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon$  for all  $k \geq N(x)$  and hence,  $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon \leq \sup_n \inf_{k \geq n} \|T_k\| + \varepsilon$  for all  $x \in B_X$  and  $\varepsilon > 0$ . So, we have  $\|T\| \leq \liminf_k \|T_k\|$  as desired.  $\square$

**Corollary 10.4.** *Every weakly convergent sequence in a normed space must be bounded.*

*Proof.* Let  $(x_n)$  be a weakly convergent sequence in a normed space  $X$ . If we let  $Q : X \rightarrow X^{**}$  be the canonical isometry, then  $(Qx_n)$  is a bounded sequence in  $X^{**}$ . Notice that  $(x_n)$  is weakly convergent if and only if  $(Qx_n)$  is  $w^*$ -convergent. So,  $(Qx_n(x^*))$  is bounded for all  $x^* \in X^*$ . Notice that the dual space  $X^*$  must be complete. So, we can apply the Uniform Boundedness Theorem to see that  $(Qx_n)$  is bounded and so is  $(x_n)$ .  $\square$

## 11. GEOMETRY OF HILBERT SPACE I

From now on, all vector spaces are over the complex field. Recall that an *inner product* on a vector space  $V$  is a function  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$  which satisfies the following conditions.

- (i)  $(x, x) \geq 0$  for all  $x \in V$  and  $(x, x) = 0$  if and only if  $x = 0$ .
- (ii)  $\overline{(x, y)} = (y, x)$  for all  $x, y \in V$ .
- (iii)  $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$  for all  $x, y, z \in V$  and  $\alpha, \beta \in \mathbb{C}$ .

Consequently, for each  $x \in V$ , the map  $y \in V \mapsto (x, y) \in \mathbb{C}$  is conjugate linear by the conditions (ii) and (iii), that is  $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$  for all  $y, z \in V$  and  $\alpha, \beta \in \mathbb{C}$ .

Also, the inner product  $(\cdot, \cdot)$  will give a norm on  $V$  which is defined by

$$\|x\| := \sqrt{(x, x)}$$

for  $x \in V$ .

We first recall the following useful properties of an inner product space which can be found in the standard text books of linear algebras.

**Proposition 11.1.** *Let  $V$  be an inner product space. For all  $x, y \in V$ , we always have:*

- (i): **(Cauchy-Schwarz inequality):**  $|(x, y)| \leq \|x\|\|y\|$  Consequently, the inner product on  $V \times V$  is jointly continuous.
- (ii): **(Parallelogram law):**  $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Furthermore, a norm  $\|\cdot\|$  on a vector space  $X$  is induced by an inner product if and only if it satisfies the Parallelogram law. In this case such inner product is given by the following:

$$\operatorname{Re}(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \quad \text{and} \quad \operatorname{Im}(x, y) = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

for all  $x, y \in X$ .

**Example 11.2.** *It follows from Proposition 11.1 immediately that  $\ell^2$  is a Hilbert space and  $\ell^p$  is not for all  $p \in [1, \infty] \setminus \{2\}$ .*

From now on, all vector spaces are assumed to be a complex inner product spaces. Recall that two vectors  $x$  and  $y$  in an inner product space  $V$  are said to be *orthogonal* if  $(x, y) = 0$ .

**Proposition 11.3. (Bessel's inequality) :** *Let  $\{e_1, \dots, e_N\}$  be an orthonormal set in an inner product space  $V$ , that is  $(e_i, e_j) = 1$  if  $i = j$ , otherwise is equal to 0. Then for any  $x \in V$ , we have*

$$\sum_{i=1}^N |(x, e_i)|^2 \leq \|x\|^2.$$

*Proof.* It can be obtained by the following equality immediately

$$\|x - \sum_{i=1}^N (x, e_i)e_i\|^2 = \|x\|^2 - \sum_{i=1}^N |(x, e_i)|^2.$$

□

**Corollary 11.4.** *Let  $(e_i)_{i \in I}$  be an orthonormal set in an inner product space  $V$ . Then for any element  $x \in V$ , the set*

$$\{i \in I : (e_i, x) \neq 0\}$$

*is countable.*

*Proof.* Note that for each  $x \in V$ , we have

$$\{i \in I : (e_i, x) \neq 0\} = \bigcup_{n=1}^{\infty} \{i \in I : |(e_i, x)| \geq 1/n\}.$$

Then the Bessel's inequality implies that the set  $\{i \in I : |(e_i, x)| \geq 1/n\}$  must be finite for each  $n \geq 1$ . So the result follows.  $\square$

The following is one of the most important classes in mathematics.

**Definition 11.5.** A Hilbert space is a Banach space whose norm is given by an inner product.

In the rest of this section,  $X$  always denotes a complex Hilbert space with an inner product  $(\cdot, \cdot)$ .

**Proposition 11.6.** Let  $(e_n)$  be a sequence of orthonormal vectors in a Hilbert space  $X$ . Then for any  $x \in V$ , the series  $\sum_{n=1}^{\infty} (x, e_n)e_n$  is convergent.

Moreover, if  $(e_{\sigma(n)})$  is a rearrangement of  $(e_n)$ , that is,  $\sigma : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$  is a bijection. Then we have

$$\sum_{n=1}^{\infty} (x, e_n)e_n = \sum_{n=1}^{\infty} (x, e_{\sigma(n)})e_{\sigma(n)}.$$

*Proof.* Since  $X$  is a Hilbert space, the convergence of the series  $\sum_{n=1}^{\infty} (x, e_n)e_n$  follows from the Bessel's inequality at once. In fact, if we put  $s_p := \sum_{n=1}^p (x, e_n)e_n$ , then we have

$$\|s_{p+k} - s_p\|^2 = \sum_{p+1 \leq n \leq p+k} |(x, e_n)|^2.$$

Now put  $y = \sum_{n=1}^{\infty} (x, e_n)e_n$  and  $z = \sum_{n=1}^{\infty} (x, e_{\sigma(n)})e_{\sigma(n)}$ . Notice that we have

$$\begin{aligned} (y, y - z) &= \lim_N \left( \sum_{n=1}^N (x, e_n)e_n, \sum_{n=1}^N (x, e_n)e_n - z \right) \\ &= \lim_N \sum_{n=1}^N |(x, e_n)|^2 - \lim_N \sum_{n=1}^N (x, e_n) \sum_{j=1}^{\infty} \overline{(x, e_{\sigma(j)})} (e_n, e_{\sigma(j)}) \\ &= \sum_{n=1}^{\infty} |(x, e_n)|^2 - \lim_N \sum_{n=1}^N (x, e_n) \overline{(x, e_n)} \quad (\text{N.B: for each } n, \text{ there is a unique } j \text{ such that } n = \sigma(j)) \\ &= 0. \end{aligned}$$

Similarly, we have  $(z, y - z) = 0$ . The result follows.  $\square$

A family of an orthonormal vectors, say  $\mathcal{B}$ , in  $X$  is said to be **complete** if it is maximal with respect to the set inclusion order, that is, if  $\mathcal{C}$  is another family of orthonormal vectors with  $\mathcal{B} \subseteq \mathcal{C}$ , then  $\mathcal{B} = \mathcal{C}$ .

A complete orthonormal subset of  $X$  is also called an **orthonormal base** of  $X$ .

**Proposition 11.7.** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $X$ . Then the followings are equivalent:

- (i):  $\{e_i\}_{i \in I}$  is complete;
- (ii): if  $(x, e_i) = 0$  for all  $i \in I$ , then  $x = 0$ ;
- (iii): for any  $x \in X$ , we have  $x = \sum_{i \in I} (x, e_i)e_i$ ;

(iv): for any  $x \in X$ , we have  $\|x\|^2 = \sum_{i \in I} |(x, e_i)|^2$ .

In this case, the expression of each element  $x \in X$  in Part (iii) is unique.

**Note :** there are only countable many  $(x, e_i) \neq 0$  by Corollary 11.4, so the sums in (iii) and (iv) are convergent by Proposition 11.6.

**Proposition 11.8.** *Let  $X$  be a Hilbert space. Then*

(i) :  $X$  possesses an orthonormal base.

(ii) : If  $\{e_i\}_{i \in I}$  and  $\{f_j\}_{j \in J}$  both are the orthonormal bases for  $X$ , then  $I$  and  $J$  have the same cardinality. In this case, the cardinality  $|I|$  of  $I$  is called the orthonormal dimension of  $X$ .

*Proof.* Part (i) follows from Zorn's Lemma at once.

For part (ii), if the cardinality  $|I|$  is finite, then the assertion is clear since  $|I| = \dim X$  (vector space dimension) in this case.

Now assume that  $|I|$  is infinite, for each  $e_i$ , put  $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$ . Note that since  $\{e_i\}_{i \in I}$  is maximal, Proposition 11.7 implies that we have

$$\{f_j\}_{j \in J} \subseteq \bigcup_{i \in I} J_{e_i}.$$

Notice that  $J_{e_i}$  is countable for each  $e_i$  by using Proposition 11.4. On the other hand, we have  $|\mathbb{N}| \leq |I|$  because  $|I|$  is infinite and thus  $|\mathbb{N} \times I| = |I|$ . Then we have

$$|J| \leq \sum_{i \in I} |J_{e_i}| = \sum_{i \in I} |\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

From symmetry argument, we also have  $|I| \leq |J|$ . □

**Remark 11.9.** *Recall that a vector space dimension of  $X$  is defined by the cardinality of a maximal linearly independent set in  $X$ .*

*Notice that if  $X$  is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.*

*Also, the vector space dimension is larger than the orthonormal dimension in general since every orthogonal set must be linearly independent.*

We say that two Hilbert spaces  $X$  and  $Y$  are said to be *isomorphic* if there is linear isomorphism  $U$  from  $X$  onto  $Y$  such that  $(Ux, Ux') = (x, x')$  for all  $x, x' \in X$ . In this case  $U$  is called a *unitary operator*.

**Theorem 11.10.** *Two Hilbert spaces are isomorphic if and only if they have the same orthonormal dimension.*

*Proof.* The converse part ( $\Leftarrow$ ) is clear.

Now for the ( $\Rightarrow$ ) part, let  $X$  and  $Y$  be isomorphic Hilbert spaces. Let  $U : X \rightarrow Y$  be a unitary. Note that if  $\{e_i\}_{i \in I}$  is an orthonormal base of  $X$ , then  $\{Ue_i\}_{i \in I}$  is also an orthonormal base of  $Y$ . Thus the necessary part follows from Proposition 11.8 at once. □

**Corollary 11.11.** *Every separable Hilbert space is isomorphic to  $\ell^2$  or  $\mathbb{C}^n$  for some  $n$ .*

*Proof.* Let  $X$  be a separable Hilbert space.

If  $\dim X < \infty$ , then it is clear that  $X$  is isomorphic to  $\mathbb{C}^n$  for  $n = \dim X$ .

Now suppose that  $\dim X = \infty$  and its orthonormal dimension is larger than  $|\mathbb{N}|$ , that is  $X$  has an orthonormal base  $\{f_i\}_{i \in I}$  with  $|I| > |\mathbb{N}|$ . Note that since  $\|f_i - f_j\| = \sqrt{2}$  for all  $i, j \in I$  with  $i \neq j$ . This implies that  $B(f_i, 1/4) \cap B(f_j, 1/4) = \emptyset$  for  $i \neq j$ .

On the other hand, if we let  $D$  be a countable dense subset of  $X$ , then  $B(f_i, 1/4) \cap D \neq \emptyset$  for all  $i \in I$ . So for each  $i \in I$ , we can pick up an element  $x_i \in D \cap B(f_i, 1/4)$ . Therefore, one can define an injection from  $I$  into  $D$ . It is absurd to the countability of  $D$ .  $\square$

## 12. GEOMETRY OF HILBERT SPACE II

In this section, let  $X$  always denote a complex Hilbert space.

**Proposition 12.1.** *If  $D$  is a closed convex subset of  $X$ , then there is a unique element  $z \in D$  such that*

$$\|z\| = \inf\{\|x\| : x \in D\}.$$

*Consequently, for any element  $u \in X$ , there is a unique element  $w \in D$  such that*

$$\|u - w\| = d(u, D) := \inf\{\|u - x\| : x \in D\}.$$

*Proof.* We first claim the existence of such  $z$ .

Let  $d := \inf\{\|x\| : x \in D\}$ . Then there is a sequence  $(x_n)$  in  $D$  such that  $\|x_n\| \rightarrow d$ . Notice that  $(x_n)$  is a Cauchy sequence. In fact, the Parallelogram Law implies that

$$\left\|\frac{x_m - x_n}{2}\right\|^2 = \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - \left\|\frac{x_m + x_n}{2}\right\|^2 \leq \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - d^2 \rightarrow 0$$

as  $m, n \rightarrow \infty$ , where the last inequality holds because  $D$  is convex and hence  $\frac{1}{2}(x_m + x_n) \in D$ . Let  $z := \lim_n x_n$ . Then  $\|z\| = d$  and  $z \in D$  because  $D$  is closed.

For the uniqueness, let  $z, z' \in D$  such that  $\|z\| = \|z'\| = d$ . Thanks to the Parallelogram Law again, we have

$$\left\|\frac{z - z'}{2}\right\|^2 = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - \left\|\frac{z + z'}{2}\right\|^2 \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - d^2 = 0.$$

Therefore  $z = z'$ .

The last assertion follows by considering the closed convex set  $u - D := \{u - x : x \in D\}$  immediately.  $\square$

**Proposition 12.2.** *Suppose that  $M$  is a closed subspace. Let  $u \in X$  and  $w \in M$ . Then the followings are equivalent:*

- (i):  $\|u - w\| = d(u, M)$ ;
- (ii):  $u - w \perp M$ , that is  $(u - w, x) = 0$  for all  $x \in M$ .

*Consequently, for each element  $u \in X$ , there is a unique element  $w \in M$  such that  $u - w \perp M$ .*

*Proof.* Let  $d := d(u, M)$ .

For proving (i)  $\Rightarrow$  (ii), fix an element  $x \in M$ . Then for any  $t > 0$ , note that since  $w + tx \in M$ , we have

$$d^2 \leq \|u - w - tx\|^2 = \|u - w\|^2 + \|tx\|^2 - 2\operatorname{Re}(u - w, tx) = d^2 + \|tx\|^2 - 2\operatorname{Re}(u - w, tx).$$

This implies that

$$(12.1) \quad 2\operatorname{Re}(u - w, x) \leq t\|x\|^2$$

for all  $t > 0$  and for all  $x \in M$ . So by considering  $-x$  in Eq.12.1, we obtain

$$2|\operatorname{Re}(u - w, x)| \leq t\|x\|^2.$$

for all  $t > 0$ . This implies that  $\operatorname{Re}(u - w, x) = 0$  for all  $x \in M$ . Similarly, putting  $\pm ix$  into Eq.12.1, we have  $\operatorname{Im}(u - w, x) = 0$ . So (ii) follows.

For (ii)  $\Rightarrow$  (i), we need to show that  $\|u - w\|^2 \leq \|u - x\|^2$  for all  $x \in M$ . Note that since  $u - w \perp M$  and  $w \in M$ , we have  $u - w \perp w - x$  for all  $x \in M$ . This gives

$$\|u - x\|^2 = \|(u - w) + (w - x)\|^2 = \|u - w\|^2 + \|w - x\|^2 \geq \|u - w\|^2.$$

Part (i) follows.

The last statement is obtained by Proposition 12.1 immediately.  $\square$

**Theorem 12.3.** *Let  $M$  be a closed subspace. Put*

$$M^\perp := \{x \in X : x \perp M\}.$$

*Then  $M^\perp$  is a closed subspace and we have  $X = M \oplus M^\perp$ .*

*In this case,  $M^\perp$  is called the orthogonal complement of  $M$ .*

*Proof.* It is clear that  $M^\perp$  is a closed subspace and  $M \cap M^\perp = (0)$ . It remains to show  $X = M + M^\perp$ . Let  $u \in X$ . Then by Proposition 12.2, we can find an element  $w \in M$  such that  $u - w \perp M$ . Thus  $u - w \in M^\perp$  and  $u = w + (u - w)$ . The proof is finished.  $\square$

**Corollary 12.4.** *With the notation as above, an element  $x_0 \notin M$  if and only if there is an element  $m \in M$  such that  $x_0 - m \perp M$ .*

*Proof.* It is clear from Theorem 12.3.  $\square$

**Corollary 12.5.** *If  $M$  is a closed subspace of  $X$ , then  $M^{\perp\perp} = M$ .*

*Proof.* It is clear that  $M \subseteq M^{\perp\perp}$  by the definition of  $M^{\perp\perp}$ . Now if there is  $x \in M^{\perp\perp} \setminus M$ , then by the decomposition  $X = M \oplus M^\perp$  obtained in Theorem 12.3, we have  $x = y + z$  for some  $y \in M$  and  $z \in M^\perp$ . This implies that  $z = x - y \in M^\perp \cap M^{\perp\perp} = (0)$ . This gives  $x = y \in M$ . It leads to a contradiction.  $\square$

**Remark 12.6.** *It is worthwhile pointing out that for a general Banach space  $X$  and a closed subspace  $M$  of  $X$ , it **May Not** have a complementary **Closed** subspace  $N$  of  $M$ , that is  $X = M \oplus N$ . If  $M$  has a complementary closed subspace  $X$ , we say that  $M$  is complemented in  $X$ .*

**Example 12.7.** *If  $M$  is a finite dimensional subspace of a normed space  $X$ , then  $M$  is complemented in  $X$ .*

*In fact, if  $M$  is spanned by  $\{e_i : i = 1, 2, \dots, m\}$ , then  $M$  is closed and by the Hahn-Banach Theorem, for each  $i = 1, \dots, m$ , there is  $e_i^* \in X^*$  such that  $e_i^*(e_j) = 1$  if  $i = j$ , otherwise, it is equal to 0. Put  $N := \bigcap_{i=1}^m \ker e_i^*$ . Then  $X = M \oplus N$ .*

*The following example can be found in [1].*

**Example 12.8.**  $c_0$  is not complemented in  $\ell^\infty$ .

*Proof.* It will be shown by the contradiction. Suppose that  $c_0$  is complemented in  $\ell^\infty$ .

**Claim 1:** There is a sequence  $(f_n)$  in  $(\ell^\infty)^*$  such that  $c_0 = \bigcap_{n=1}^\infty \ker f_n$ .

In fact, by the assumption, there is a closed subspace  $F$  of  $\ell^\infty$  such that  $\ell^\infty = c_0 \oplus F$ . If we let  $P$  be the projection from  $\ell^\infty$  onto  $F$  along this decomposition, then  $\ker P = c_0$  and  $P$  is bounded by the Closed Graph Theorem. Let  $e_n^* : \ell^\infty \rightarrow \mathbb{K}$  be the  $n$ -th coordinate functional. Then  $e_n^* \in (\ell^\infty)^*$ . So, if we put  $f_n = e_n^* \circ P$ , then  $f_n \in (\ell^\infty)^*$  and  $c_0 = \bigcap_{n=1}^\infty \ker f_n$  as desired.

**Claim 2:** For each irrational number  $\alpha \in [0, 1]$ , there is an infinite subset  $N_\alpha$  of  $\mathbb{N}$  such that  $N_\alpha \cap N_\beta$  is a finite set if  $\alpha$  and  $\beta$  both are distinct irrational numbers in  $[0, 1]$ .

In fact, we write  $[0, 1] \cap \mathbb{Q}$  as a sequence  $(r_n)$ . Then for each irrational  $\alpha$  in  $[0, 1]$ , there is a subsequence  $(r_{n_k})$  of  $(r_n)$  such that  $\lim_k r_{n_k} = \alpha$ . Let  $N_\alpha := \{n_k : k = 1, 2, \dots\}$ . From this, we see that  $N_\alpha \cap N_\beta$  is a finite set whenever  $\alpha, \beta \in [0, 1] \cap \mathbb{Q}^c$  with  $\alpha \neq \beta$ . Claim 2 follows.

Now for each  $\alpha \in [0, 1] \cap \mathbb{Q}^c$ , define an element  $x_\alpha \in \ell^\infty$  by  $x_\alpha(k) \equiv 1$  as  $k \in N_\alpha$ ; otherwise,  $x_\alpha(k) \equiv 0$ .

**Claim 3:** If  $f \in (\ell^\infty)^*$  with  $c_0 \subseteq \ker f$ , then for any  $\eta > 0$ , the set  $\{\alpha \in [0, 1] \cap \mathbb{Q}^c : |f(x_\alpha)| \geq \eta\}$  is finite.

Notice that by considering the decomposition  $f = \operatorname{Re}(f) + i\operatorname{Im}(f)$ , it suffices to show that the set  $\{\alpha \in [0, 1] \cap \mathbb{Q}^c : f(x_\alpha) \geq \eta\}$  is finite. Let  $\alpha_1, \dots, \alpha_N$  in  $[0, 1] \cap \mathbb{Q}^c$  such that  $f(x_{\alpha_j}) \geq \eta$ ,  $j = 1, \dots, N$ . Now for each  $j = 1, \dots, N$ , set  $y_j(k) \equiv 1$  as  $k \in N_{\alpha_j} \setminus \bigcup_{m \neq j} N_{\alpha_m}$ ; otherwise  $y_j \equiv 0$ . Notice that  $x_{\alpha_j} - y_j \in c_0$  since  $N_\alpha \cap N_\beta$  is finite for  $\alpha \neq \beta$  by Claim 2. Hence, we have  $f(x_{\alpha_j}) = f(y_j)$  for all  $j = 1, \dots, N$ . Moreover, we have  $\{k : y_j(k) = 1\} \cap \{k : y_i(k) = 1\} = \emptyset$  for  $i, j = 1, \dots, N$  with  $i \neq j$ . So, we have  $\|y\|_\infty = 1$ . Now we can conclude that

$$\|f\| \geq f\left(\sum_{j=1}^N y_j\right) = \sum_{j=1}^N f(x_{\alpha_j}) \geq N\eta.$$

This implies that  $|\{\alpha : f(\alpha) \geq \eta\}| \leq \|f\|/\eta$ . Claim 3 follows.

We can now go to complete the proof. Now let  $(f_n)$  be the sequence in  $(\ell^\infty)^*$  as found in the Claim 1. Claim 3 implies that the set  $S := \bigcup_{n=1}^\infty \{\alpha \in \mathbb{Q}^c \cap [0, 1] : f_n(x_\alpha) \neq 0\}$  is countable. Thus, there must exist  $\gamma \in [0, 1] \cap \mathbb{Q}^c$  such that  $\gamma \notin S$ . Thus, we have  $x_\gamma \in \bigcap_{n=1}^\infty \ker f_n$ . On the other hand, since  $N_\gamma$  is an infinite set, we see that  $x_\gamma \notin c_0$ . Therefore, we have  $c_0 \subsetneq \bigcap \ker f_k$  which contradicts to Claim 1.

The proof is finished.  $\square$

**Theorem 12.9. Riesz Representation Theorem :** For each  $f \in X^*$ , then there is a unique element  $v_f \in X$  such that

$$f(x) = (x, v_f)$$

for all  $x \in X$  and we have  $\|f\| = \|v_f\|$ .

Furthermore, if  $(e_i)_{i \in I}$  is an orthonormal base of  $X$ , then  $v_f = \sum_i \overline{f(e_i)} e_i$ .

*Proof.* We first prove the uniqueness of  $v_f$ . If  $z \in X$  also satisfies the condition:  $f(x) = (x, z)$  for all  $x \in X$ . This implies that  $(x, z - v_f) = 0$  for all  $x \in X$ . So  $z - v_f = 0$ .

Now for proving the existence of  $v_f$ , it suffices to show the case  $\|f\| = 1$ . Then  $\ker f$  is a closed proper subspace. Then by the orthogonal decomposition again, we have

$$X = \ker f \oplus (\ker f)^\perp.$$

Since  $f \neq 0$ , we have  $(\ker f)^\perp$  is linear isomorphic to  $\mathbb{C}$ . Also note that the restriction of  $f$  on  $(\ker f)^\perp$  is of norm one. Hence there is an element  $v_f \in (\ker f)^\perp$  with  $\|v_f\| = 1$  such that  $f(v_f) = \|f|_{(\ker f)^\perp}\| = 1$  and  $(\ker f)^\perp = \mathbb{C}v_f$ . So for each element  $x \in X$ , we have  $x = z + \alpha v_f$  for some  $z \in \ker f$  and  $\alpha \in \mathbb{C}$ . Then  $f(x) = \alpha f(v_f) = \alpha = (x, v_f)$  for all  $x \in X$ .

Concerning about the last assertion, if we put  $v_f = \sum_{i \in I} \alpha_i e_i$ , then  $f(e_j) = (e_j, v_f) = \overline{\alpha_j}$  for all  $j \in I$ . The proof is finished.  $\square$

**Corollary 12.10.** With the notation as in Theorem 12.9, Define the map

$$(12.2) \quad \Phi : f \in X^* \mapsto v_f \in X, \text{ i.e., } f(y) = (y, \Phi(f))$$

for all  $y \in X$  and  $f \in X^*$ .

And if we define  $(f, g)_{X^*} := (v_g, v_f)_X$  for  $f, g \in X^*$ . Then  $(X^*, (\cdot, \cdot)_{X^*})$  becomes a Hilbert space. Moreover,  $\Phi$  is an anti-unitary operator from  $X^*$  onto  $X$ , that is  $\Phi$  satisfies the conditions:

$$\Phi(\alpha f + \beta g) = \overline{\alpha} \Phi(f) + \overline{\beta} \Phi(g) \quad \text{and} \quad (\Phi f, \Phi g)_X = (g, f)_{X^*}$$

for all  $f, g \in X^*$  and  $\alpha, \beta \in \mathbb{C}$ .

Furthermore, if we define  $J : x \in X \mapsto f_x \in X^*$ , where  $f_x(y) := (y, x)$ , then  $J$  is the inverse of  $\Phi$ , and hence,  $J$  is an isometric conjugate linear isomorphism.



*Proof.* The result follows immediately from the observation that  $v_{f+g} = v_f + v_g$  and  $v_{\alpha f} = \bar{\alpha}v_f$  for all  $f \in X^*$  and  $\alpha \in \mathbb{C}$ .

The last assertion is clearly obtained by the Eq.12.2 above.  $\square$

**Corollary 12.11.** *Every Hilbert space is reflexive.*

*Proof.* Using the notation as in the Riesz Representation Theorem 12.9, let  $X$  be a Hilbert space. and  $Q : X \rightarrow X^{**}$  the canonical isometry. Let  $\psi \in X^{**}$ . To apply the Riesz Theorem on the dual space  $X^*$ , there exists an element  $x_0^* \in X^*$  such that

$$\psi(f) = (f, x_0^*)_{X^*}$$

for all  $f \in X^*$ . By using Corollary 12.10, there is an element  $x_0 \in X$  such that  $x_0 = v_{x_0^*}$  and thus, we have

$$\psi(f) = (f, x_0^*)_{X^*} = (x_0, v_f)_X = f(x_0)$$

for all  $f \in X^*$ . Therefore,  $\psi = Q(x_0)$  and so,  $X$  is reflexive.

The proof is finished.  $\square$

**Theorem 12.12.** *Every bounded sequence in a Hilbert space has a weakly convergent subsequence.*

*Proof.* Let  $(x_n)$  be a bounded sequence in a Hilbert space  $X$  and  $M$  be the closed subspace of  $X$  spanned by  $\{x_m : m = 1, 2, \dots\}$ . Then  $M$  is a separable Hilbert space.

**Method I :** Define a map by  $j_M : x \in M \mapsto j_M(x) := (\cdot, x) \in M^*$ . Then  $(j_M(x_n))$  is a bounded sequence in  $M^*$ . By Banach's result, Proposition 7.9,  $(j_M(x_n))$  has a  $w^*$ -convergent subsequence  $(j_M(x_{n_k}))$ . Put  $j_M(x_{n_k}) \xrightarrow{w^*} f \in M^*$ , that is  $j_M(x_{n_k})(z) \rightarrow f(z)$  for all  $z \in M$ . The Riesz Representation will assure that there is a unique element  $m \in M$  such that  $j_M(m) = f$ . So we have  $(z, x_{n_k}) \rightarrow (z, m)$  for all  $z \in M$ . In particular, if we consider the orthogonal decomposition  $X = M \oplus M^\perp$ , then  $(x, x_{n_k}) \rightarrow (x, m)$  for all  $x \in X$  and thus  $(x_{n_k}, x) \rightarrow (m, x)$  for all  $x \in X$ . Then  $x_{n_k} \rightarrow m$  weakly in  $X$  by using the Riesz Representation Theorem again.

**Method II :** We first note that since  $M$  is a separable Hilbert space, the second dual  $M^{**}$  is also separable by the reflexivity of  $M$ . So the dual space  $M^*$  is also separable (see Proposition 5.7). Let  $Q : M \rightarrow M^{**}$  be the natural canonical mapping. To apply the Banach's result Proposition 7.9 for  $X^*$ , then  $Q(x_n)$  has a  $w^*$ -convergent subsequence, says  $Q(x_{n_k})$ . This gives an element  $m \in M$  such that  $Q(m) = w^*\text{-}\lim_k Q(x_{n_k})$  because  $M$  is reflexive. So we have  $f(x_{n_k}) = Q(x_{n_k})(f) \rightarrow Q(m)(f) = f(m)$  for all  $f \in M^*$ . Using the same argument as in **Method I** again,  $x_{n_k}$  weakly converges to  $m$  as desired.  $\square$

**Remark 12.13.** *It is well known that we have the following Theorem due to R. C. James (the proof is highly non-trivial):*

*A normed space  $X$  is reflexive if and only if every bounded sequence in  $X$  has a weakly convergent subsequence.*

*Theorem 12.12 can be obtained by the James's Theorem directly. However, Theorem 12.12 gives a simple proof in the Hilbert spaces case.*

### 13. OPERATORS ON A HILBERT SPACE

Throughout this section, all spaces are complex Hilbert spaces. Let  $B(X, Y)$  denote the space of all bounded linear operators from  $X$  into  $Y$ . If  $X = Y$ , write  $B(X)$  for  $B(X, X)$ .

Let  $T \in B(X, Y)$ . We will make use the following simple observation:

$$(13.1) \quad (Tx, y) = 0 \text{ for all } x \in X; y \in Y \quad \text{if and only if} \quad T = 0.$$

Therefore, the elements in  $B(X, Y)$  are uniquely determined by the Eq.13.1, that is,  $T = S$  in  $B(X, Y)$  if and only if  $(Tx, y) = (Sx, y)$  for all  $x \in X$  and  $y \in Y$ .

**Remark 13.1.** For Hilbert spaces  $H_1$  and  $H_2$ , we consider their direct sum  $H := H_1 \oplus H_2$ . If we define the inner product on  $H$  by

$$(x_1 \oplus x_2, y_1 \oplus y_2) := (x_1, y_1)_{H_1} + (x_2, y_2)_{H_2}$$

for  $x_1 \oplus x_2$  and  $y_1 \oplus y_2$  in  $H$ , then  $H$  becomes a Hilbert space. Now for each  $T \in B(H_1, H_2)$ , we can define an element  $\tilde{T} \in B(H)$  by  $\tilde{T}(x_1 \oplus x_2) := 0 \oplus Tx_1$ . So, the space  $B(H_1, H_2)$  can be viewed as a closed subspace of  $B(H)$ . Thus, we can consider the case of  $H_1 = H_2$  for studying the space  $B(H_1, H_2)$ .

**Proposition 13.2.** Let  $T \in B(X)$ . Then we have

- (i):  $T = 0$  if and only if  $(Tx, x) = 0$  for all  $x \in X$ . Consequently, for  $T, S \in B(X)$ ,  $T = S$  if and only if  $(Tx, x) = (Sx, x)$  for all  $x \in X$ .
- (ii):  $\|T\| = \sup\{|(Tx, y)| : x, y \in X \text{ with } \|x\| = \|y\| = 1\}$ .

*Proof.* It is clear that the necessary part in Part (i). Now we are going to the sufficient part in Part (i), that is we assume that  $(Tx, x) = 0$  for all  $x \in X$ . This implies that we have

$$0 = (T(x + iy), x + iy) = (Tx, x) + i(Ty, x) - i(Tx, y) + (Tiy, iy) = i(Ty, x) - i(Tx, y).$$

So we have  $(Ty, x) - (Tx, y) = 0$  for all  $x, y \in X$ . In particular, if we replace  $y$  by  $iy$  in the equation, then we get  $i(Ty, x) - \bar{i}(Tx, y) = 0$  and hence we have  $(Ty, x) + (Tx, y) = 0$ . Therefore we have  $(Tx, y) = 0$ .

For part (ii) : Let  $\alpha = \sup\{|(Tx, y)| : x, y \in X \text{ with } \|x\| = \|y\| = 1\}$ . It is clear that we have  $\|T\| \geq \alpha$ . It needs to show  $\|T\| \leq \alpha$ .

In fact, for each  $x \in X$  with  $\|x\| = 1$ , then by the Hahn-Banach Theorem, there is  $f \in X^*$  with  $\|f\| = 1$  such that  $f(Tx) = \|Tx\|$ . The Riesz Representation Theorem, we can find an element  $y_f \in X$  with  $\|y_f\| = \|f\| = 1$  so that we have  $\|Tx\| = f(Tx) = (x, y_f) \leq \alpha$  for all  $x \in X$  with  $\|x\| = 1$ . This implies that  $\|T\| \leq \alpha$ . The proof is finished.  $\square$

**Proposition 13.3.** Let  $T \in B(X)$ . Then there is a unique element  $T^*$  in  $B(X)$  such that

$$(13.2) \quad (Tx, y) = (x, T^*y)$$

In this case,  $T^*$  is called the adjoint operator of  $T$ .

*Proof.* We first show the uniqueness. Suppose that there are  $S_1, S_2$  in  $B(X)$  which satisfy the Eq.13.2. Then  $(x, S_1y) = (x, S_2y)$  for all  $x, y \in X$ . Eq.13.1 implies that  $S_1 = S_2$ .

Finally, we prove the existence. Note that if we fix an element  $y \in X$ , define the map  $f_y(x) := (Tx, y)$  for all  $x \in X$ . Then  $f_y \in X^*$ . The Riesz Representation Theorem implies that there is a unique element  $y^* \in X$  such that  $(Tx, y) = (x, y^*)$  for all  $x \in X$  and  $\|f_y\| = \|y^*\|$ . On the other hand, we have

$$|f_y(x)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$$

for all  $x, y \in X$  and thus  $\|f_y\| \leq \|T\|\|y\|$ . If we put  $T^*(y) := y^*$ , then  $T^*$  satisfies the Eq.13.2. Also, we have  $\|T^*y\| = \|y^*\| = \|f_y\| \leq \|T\|\|y\|$  for all  $y \in X$ . So  $T^* \in B(X)$  with  $\|T^*\| \leq \|T\|$  indeed. Hence  $T^*$  is as desired.  $\square$

**Remark 13.4.** Let  $S, T : X \rightarrow X$  be linear operators (without assuming to be bounded). If they satisfy the Eq.13.2 above, i.e.,

$$(Tx, y) = (x, Sy)$$

for all  $x, y \in X$ . Using the Closed Graph Theorem, one can show that  $S$  and  $T$  both are automatically bounded.

In fact, let  $(x_n)$  be a sequence in  $X$  such that  $\lim x_n = x$  and  $\lim Sx_n = y$  for some  $x, y \in X$ . Now for any  $z \in X$ , we have

$$(z, Sx) = (Tz, x) = \lim(Tz, x_n) = \lim(z, Sx_n) = (z, y).$$

Thus  $Sx = y$  and hence  $S$  is bounded by the Closed Graph Theorem.

Similarly, we can also see that  $T$  is bounded.

**Remark 13.5.** Let  $T \in B(X)$ . Let  $T^t : X^* \rightarrow X^*$  be the transpose of  $T$  which is defined by  $T^t(f) := f \circ T \in X^*$  for  $f \in X^*$  (see Proposition 5.9). Then we have the following commutative diagram (**Check!**)

$$\begin{array}{ccc} X & \xrightarrow{T^*} & X \\ J_X \downarrow & & \downarrow J_X \\ X^* & \xrightarrow{T^t} & X^* \end{array}$$

where  $J_X : X \rightarrow X^*$  is the anti-unitary given by the Riesz Representation Theorem (see Corollary 12.10).

**Proposition 13.6.** Let  $T, S \in B(X)$ . Then we have

(i):  $T^* \in B(X)$  and  $\|T^*\| = \|T\|$ .

(ii): The map  $T \in B(X) \mapsto T^* \in B(X)$  is an isometric conjugate anti-isomorphism, that is,

$$(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^* \quad \text{for all } \alpha, \beta \in \mathbb{C}; \quad \text{and} \quad (TS)^* = S^* T^*.$$

(iii):  $\|T^* T\| = \|T\|^2$ .

*Proof.* For Part (i), in the proof of Proposition 13.3, we have shown that  $\|T^*\| \leq \|T\|$ . And the reverse inequality clearly follows from  $T^{**} = T$ .

The Part (ii) follows from the adjoint operators are uniquely determined by the Eq.13.2 above.

For Part (iii), we always have  $\|T^* T\| \leq \|T^*\| \|T\| = \|T\|^2$ . For the reverse inequality, let  $x \in B_X$ . Then

$$\|Tx\|^2 = (Tx, Tx) = (T^* Tx, x) \leq \|T^* Tx\| \|x\| \leq \|T^* T\| \|x\|^2.$$

Therefore, we have  $\|T\|^2 \leq \|T^* T\|$ . □

**Example 13.7.** If  $X = \mathbb{C}^n$  and  $D = (a_{ij})_{n \times n}$  an  $n \times n$  matrix, then  $D^* = (\overline{a_{ji}})_{n \times n}$ . In fact, notice that

$$a_{ji} = (De_i, e_j) = (e_i, D^* e_j) = \overline{(D^* e_j, e_i)}.$$

So if we put  $D^* = (d_{ij})_{n \times n}$ , then  $d_{ij} = (D^* e_j, e_i) = \overline{a_{ji}}$ .

**Example 13.8.** Let  $\ell^2(\mathbb{N}) := \{x : \mathbb{N} \rightarrow \mathbb{C} : \sum_{i=0}^{\infty} |x(i)|^2 < \infty\}$ . And put  $(x, y) := \sum_{i=0}^{\infty} x(i) \overline{y(i)}$ .

Define the operator  $D \in B(\ell^2(\mathbb{N}))$  (called the unilateral shift) by

$$Dx(i) = x(i-1)$$

for  $i \in \mathbb{N}$  and where we set  $x(-1) := 0$ , that is  $D(x(0), x(1), \dots) = (0, x(0), x(1), \dots)$ . Then  $D$  is an isometry and the adjoint operator  $D^*$  is given by

$$D^*x(i) := x(i+1)$$

for  $i = 0, 1, \dots$ , that is  $D^*(x(0), x(1), \dots) = (x(1), x(2), \dots)$ .

Indeed one can directly check that

$$(Dx, y) = \sum_{i=0}^{\infty} x(i-1)\overline{y(i)} = \sum_{j=0}^{\infty} x(j)\overline{y(j+1)} = (x, D^*y).$$

Note that  $D^*$  is NOT an isometry.

**Example 13.9.** Let  $\ell^\infty(\mathbb{N}) = \{x : \mathbb{N} \rightarrow \mathbb{C} : \sup_{i \geq 0} |x(i)| < \infty\}$  and  $\|x\|_\infty := \sup_{i \geq 0} |x(i)|$ . For each  $x \in \ell^\infty$ , define  $M_x \in B(\ell^2(\mathbb{N}))$  by

$$M_x(\xi) := x \cdot \xi$$

for  $\xi \in \ell^2(\mathbb{N})$ , where  $(x \cdot \xi)(i) := x(i)\xi(i)$ ;  $i \in \mathbb{N}$ .

Then  $\|M_x\| = \|x\|_\infty$  and  $M_x^* = M_{\bar{x}}$ , where  $\bar{x}(i) := \overline{x(i)}$ .

**Definition 13.10.** Let  $T \in B(X)$  and let  $I$  be the identity operator on  $X$ .  $T$  is said to be

- (i) : selfadjoint if  $T^* = T$ ;
- (ii) : normal if  $T^*T = TT^*$ ;
- (iii) : unitary if  $T^*T = TT^* = I$ .

**Proposition 13.11.** We have

(i) : Let  $T : X \rightarrow X$  be a linear operator.  $T$  is selfadjoint if and only if

$$(13.3) \quad (Tx, y) = (x, Ty) \quad \text{for all } x, y \in X.$$

(ii) :  $T$  is normal if and only if  $\|Tx\| = \|T^*x\|$  for all  $x \in X$ .

*Proof.* The necessary part of Part (i) is clear.

Now suppose that the Eq.13.3 holds, it needs to show that  $T$  is bounded. Indeed, it follows from Remark13.4 at once.

For Part (ii), note that by Proposition 13.2,  $T$  is normal if and only if  $(T^*Tx, x) = (TT^*x, x)$ . So, Part (ii) follows from that

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2$$

for all  $x \in X$ . □

**Proposition 13.12.** Let  $T \in B(H)$ . We have the following assertions.

- (i) :  $T$  is selfadjoint if and only if  $(Tx, x) \in \mathbb{R}$  for all  $x \in H$ .
- (ii) : If  $T$  is selfadjoint, then  $\|T\| = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$ .

*Proof.* Part (i) is clearly follows from Proposition13.2.

For Part (ii), if we let  $a = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$ , then it is clear that  $a \leq \|T\|$ . We are now going to show the reverse inequality. Since  $T$  is selfadjoint, one can directly check that

$$(T(x+y), x+y) - (T(x-y), x-y) = 4\operatorname{Re}(Tx, y)$$

for all  $x, y \in H$ . Thus if  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and  $(Tx, y) \in \mathbb{R}$ , then by using the Parallelogram Law, we have

$$(13.4) \quad |(Tx, y)| \leq \frac{a}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{a}{2}(\|x\|^2 + \|y\|^2) = a.$$

Now for  $x, y \in H$  with  $\|x\| = \|y\| = 1$ , by considering the polar form of  $(Tx, y) = re^{i\theta}$ , the Eq.13.4 gives

$$|(Tx, y)| = |(Tx, e^{i\theta}y)| \leq a.$$

Since  $\|T\| = \sup_{\|x\|=\|y\|=1} |(Tx, y)|$ , we have  $\|T\| \leq a$  as desired. The proof is finished.  $\square$

**Proposition 13.13.** *Let  $T \in B(X)$ . Then we have*

$$\ker T = (imT^*)^\perp \quad \text{and} \quad (\ker T)^\perp = \overline{imT^*}$$

where  $imT$  denotes the image of  $T$ .

*Proof.* The first equality clearly follows from  $x \in \ker T$  if and only if  $0 = (Tx, z) = (x, T^*z)$  for all  $z \in X$ .

On the other hand, it is clear that we have  $M^\perp = \overline{M}^\perp$  for any subspace  $M$  of  $X$ . This together with the first equality and Corollary12.5 will yield the second equality at once.  $\square$

**Proposition 13.14.** *Let  $(E, \|\cdot\|)$  be a Banach space. Let  $M$  and  $N$  be the closed subspaces of  $E$  such that*

$$E = M \oplus N \quad \dots\dots\dots (*)$$

Define an operator  $Q : E \rightarrow E$  by  $Q(y + z) = y$  for  $y \in M$  and  $z \in N$ . Then  $Q$  is bounded. In this case,  $Q$  is called the projection with respect to the decomposition  $(*)$ .

Furthermore, if  $E$  is a Hilbert space, then  $N = M^\perp$  (and hence  $(*)$  is the orthogonal decomposition of  $E$  with respect to  $M$ ) if and only if  $Q$  satisfies the conditions:  $Q^2 = Q$  and  $Q^* = Q$ . And  $Q$  is called the orthogonal projection (or projection for simply) with respect to  $M$ .

*Proof.* For showing the boundedness of  $Q$ , by using the Closed Graph Theorem, we need to show that if  $(x_n)$  is a sequence in  $E$  such that  $\lim x_n = x$  and  $\lim Qx_n = u$  for some  $x, u \in E$ , then  $Qx = u$ .

Indeed, if we let  $x_n = y_n \oplus z_n$  and  $u = v \oplus w$ , where  $y_n, v \in M$  and  $z_n, w \in N$ , then  $Qx_n = y_n$ . Notice that  $(z_n)$  is a convergent sequence in  $E$  because  $z_n = x_n - y_n$  and  $(x_n)$  and  $(y_n)$  both are convergent. Let  $w = \lim z_n$ . This implies that

$$x = \lim x_n = \lim(y_n \oplus z_n) = u \oplus w.$$

Since  $M$  and  $N$  are closed, we have  $u \in M$  and  $z \in N$ . Therefore, we have  $Qx = u$  as desired.

For the last assertion, we further assume that  $E$  is a Hilbert space.

It is clear from the definition of  $Q$  that  $Q(y) = y$  and  $Q(z) = 0$  for all  $y \in M$  and  $z \in N$ . Thus we have  $Q^2 = Q$ .

Now if  $N = M^\perp$ , then for  $y, y' \in M$  and  $z, z' \in N$ , we have

$$(Q(y + z), y' + z') = (y, y') = (y + z, Q(y' + z')).$$

So  $Q^* = Q$ .

The converse of the last statement follows from Proposition 13.13 at once because  $\ker Q = N$  and  $imQ = M$ .

The proof is complete.  $\square$

**Proposition 13.15.** *When  $X$  is a Hilbert space, we put  $\mathcal{M}$  the set of all closed subspaces of  $X$  and  $\mathcal{P}$  the set of all orthogonal projections on  $X$ . Now for each  $M \in \mathcal{M}$ , let  $P_M$  be the corresponding projection with respect to the orthogonal decomposition  $X = M \oplus M^\perp$ . Then there is an one-one correspondence between  $\mathcal{M}$  and  $\mathcal{P}$  which is defined by*

$$M \in \mathcal{M} \mapsto P_M \in \mathcal{P}.$$

Furthermore, if  $M, N \in \mathcal{M}$ , then we have

- (i) :  $M \subseteq N$  if and only if  $P_M P_N = P_N P_M = P_M$ .  
(ii) :  $M \perp N$  if and only if  $P_M P_N = P_N P_M = 0$ .

*Proof.* It first follows from Proposition 13.14 that  $P_M \in \mathcal{P}$ .

Indeed the inverse of the correspondence is given by the following. If we let  $Q \in \mathcal{P}$  and  $M = Q(X)$ , then  $M$  is closed because  $M = \ker(I - Q)$  and  $I - Q$  is bounded. Also it is clear that  $X = Q(X) \oplus (I - Q)X$  with  $\ker Q = M^\perp$ . Hence  $M$  is the corresponding closed subspace of  $X$ , that is  $M \in \mathcal{M}$  and  $P_M = Q$  as desired.

For the final assertion, Part (i) and (ii) follow immediately from the orthogonal decompositions  $X = M \oplus M^\perp = N \oplus N^\perp$  and together with the clear facts that  $M \subseteq N$  if and only if  $N^\perp \subseteq M^\perp$ .  $\square$

#### 14. SPECTRAL THEORY I

**Definition 14.1.** Let  $E$  be a normed space and let  $T \in B(E)$ . The spectrum of  $T$ , write  $\sigma(T)$ , is defined by

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(E)\}.$$

**Remark 14.2.** More precise, for a normed space  $E$ , an operator  $T \in B(E)$  is said to be invertible in  $B(E)$  if  $T$  is a linear isomorphism and the inverse  $T^{-1}$  is also bounded. However, if  $E$  is complete, the Open Mapping Theorem assures that the inverse  $T^{-1}$  is bounded automatically. So if  $E$  is a Banach space and  $T \in B(E)$ , then  $\lambda \notin \sigma(T)$  if and only if  $T - \lambda := T - \lambda I$  is a linear isomorphism. So  $\lambda$  lies in the spectrum  $\sigma(T)$  if and only if  $T - \lambda$  is either not one-one or not surjective.

In particular, if there is a non-zero element  $v \in X$  such that  $Tv = \lambda v$ , then  $\lambda \in \sigma(T)$  and  $\lambda$  is called an eigenvalue of  $T$  with eigenvector  $v$ .

We also write  $\sigma_p(T)$  for the set of all eigenvalue of  $T$  and call  $\sigma_p(T)$  the point spectrum.

**Example 14.3.** Let  $E = \mathbb{C}^n$  and  $T = (a_{ij})_{n \times n} \in M_n(\mathbb{C})$ . Then  $\lambda \in \sigma(T)$  if and only if  $\lambda$  is an eigenvalue of  $T$  and thus  $\sigma(T) = \sigma_p(T)$ .

**Example 14.4.** Let  $E = (c_{00}(\mathbb{N}), \|\cdot\|_\infty)$  (note that  $c_{00}(\mathbb{N})$  is not a Banach space). Define the map  $T : c_{00}(\mathbb{N}) \rightarrow c_{00}(\mathbb{N})$  by

$$Tx(k) := \frac{x(k)}{k+1}$$

for  $x \in c_{00}(\mathbb{N})$  and  $i \in \mathbb{N}$ .

Then  $T$  is bounded, in fact,  $\|Tx\|_\infty \leq \|x\|_\infty$  for all  $x \in c_{00}(\mathbb{N})$ .

On the other hand, we note that if  $\lambda \in \mathbb{C}$  and  $x \in c_{00}(\mathbb{N})$ , then

$$(T - \lambda)x(k) = \left(\frac{1}{k+1} - \lambda\right)x(k).$$

From this we see that  $\sigma_p(T) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ . And if  $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ , then  $T - \lambda$  is a linear isomorphism and its inverse is given by

$$(T - \lambda)^{-1}x(k) = \left(\frac{1}{k+1} - \lambda\right)^{-1}x(k).$$

So,  $(T - \lambda)^{-1}$  is unbounded if  $\lambda = 0$  and thus  $0 \in \sigma(T)$ .

On the other hand, if  $\lambda \neq 0$ , then  $(T - \lambda)^{-1}$  is bounded. In fact, if  $\lambda = a + ib \neq 0$ , for  $a, b \in \mathbb{R}$ , then  $\eta := \min_k \left| \frac{1}{1+k} - a \right|^2 + |b|^2 > 0$  because  $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ . This gives

$$\|(T - \lambda)^{-1}\| = \sup_{k \in \mathbb{N}} \left| \left(\frac{1}{k+1} - \lambda\right)^{-1} \right| < \eta^{-1} < \infty.$$

It can now be concluded that  $\sigma(T) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$ .

**Proposition 14.5.** *Let  $E$  be a Banach space and  $T \in B(E)$ . Then*

- (i) :  $I - T$  is invertible in  $B(E)$  whenever  $\|T\| < 1$ .
- (ii) : If  $|\lambda| > \|T\|$ , then  $\lambda \notin \sigma(T)$ .
- (iii) :  $\sigma(T)$  is a compact subset of  $\mathbb{C}$ .
- (iv) : If we let  $GL(E)$  the set of all invertible elements in  $B(E)$ , then  $GL(E)$  is an open subset of  $B(E)$  with respect to the  $\|\cdot\|$ -topology.

*Proof.* Notice that since  $B(E)$  is complete, Part (i) clearly follows from the following equality immediately:

$$(I - T)(I + T + T^2 + \dots + T^{N-1}) = I - T^N$$

for all  $N \in \mathbb{N}$ .

For Part (ii), if  $|\lambda| > \|T\|$ , then by Part (i), we see that  $I - \frac{1}{\lambda}T$  is invertible and so is  $\lambda I - T$ . This implies  $\lambda \notin \sigma(T)$ .

For Part (iii), since  $\sigma(T)$  is bounded by Part (ii), it needs to show that  $\sigma(T)$  is closed.

Let  $c \in \mathbb{C} \setminus \sigma(T)$ . It needs to find  $r > 0$  such that  $\mu \notin \sigma(T)$  as  $|\mu - c| < r$ . Note that since  $T - c$  is invertible, then for  $\mu \in \mathbb{C}$ , we have  $T - \mu = (T - c) - (\mu - c) = (T - c)(I - (\mu - c)(T - c)^{-1})$ . Therefore, if  $\|(\mu - c)(T - c)^{-1}\| < 1$ , then  $T - \mu$  is invertible by Part (i). So if we take  $0 < r < \frac{1}{\|(T - c)^{-1}\|}$ ,

then  $r$  is as desired, that is,  $B(c, r) \subseteq \mathbb{C} \setminus \sigma(T)$ . Hence  $\sigma(T)$  is closed.

For the last assertion, let  $T \in GL(E)$ . Notice that for any  $S \in B(E)$ , we have  $\|T - S\| \leq \|T\|\|I - T^{-1}S\|$ . So if  $\|S\| < \frac{1}{\|T^{-1}\|}$ , then  $T - S$  is invertible by Part (i). Therefore we have

$$B(T, \frac{1}{\|T^{-1}\|}) \subseteq GL(E).$$

The proof is finished.  $\square$

**Corollary 14.6.** *If  $U$  is a unitary operator on a Hilbert space  $X$ , then  $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ .*

*Proof.* Since  $\|U\| = 1$ , we have  $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$  by Proposition 14.5(ii).

Now if  $|\lambda| < 1$ , then  $\|\lambda U^*\| < 1$ . By using Proposition 14.5 again, we have  $I - \lambda U^*$  is invertible. This implies that  $U - \lambda = U(I - \lambda U^*)$  is also invertible and thus  $\lambda \notin \sigma(U)$ .  $\square$

**Example 14.7.** *Let  $E = \ell^2(\mathbb{N})$  and  $D \in B(E)$  be the right unilateral shift operator as in Example 13.8. Recall that  $Dx(k) := x(k - 1)$  for  $i \in \mathbb{N}$  and  $x(-1) := 0$ . Then  $\sigma_p(D) = \emptyset$  and  $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ .*

*We first claim that  $\sigma_p(D) = \emptyset$ .*

*Suppose that  $\lambda \in \mathbb{C}$  and  $x \in \ell^2(\mathbb{N})$  satisfy the equation  $Dx = \lambda x$ . Then by the definition of  $D$ , we have*

$$x(k - 1) = \lambda x(k) \quad \dots \dots \dots (*)$$

*for all  $k \in \mathbb{N}$ .*

*If  $\lambda \neq 0$ , then we have  $x(k) = \lambda^{-1}x_{k-1}$  for all  $i \in \mathbb{N}$ . Since  $x(-1) = 0$ , this forces  $x(k) = 0$  for all  $i$ , that is  $x = 0$  in  $\ell^2(\mathbb{N})$ .*

*On the other hand if  $\lambda = 0$ , the Eq.(\*) gives  $x(k - 1) = 0$  for all  $k$  and so  $x = 0$  again.*

*Therefore  $\sigma_p(D) = \emptyset$ .*

*Finally, we are going to show  $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ .*

*Note that since  $D$  is an isometry,  $\|D\| = 1$ . Proposition 14.5 tells us that*

$$\sigma(D) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

*Notice that since  $\sigma_p(D)$  is empty, it suffices to show that  $D - \mu$  is not surjective for all  $\mu \in \mathbb{C}$  with  $|\mu| \leq 1$ .*

*Now suppose that there is  $\lambda \in \mathbb{C}$  with  $|\lambda| \leq 1$  such that  $D - \lambda$  is surjective.*

*We consider the case when  $|\lambda| = 1$  first.*

Let  $e_1 = (1, 0, 0, \dots) \in \ell^2(\mathbb{N})$ . Then by the assumption, there is  $x \in \ell^2(\mathbb{N})$  such that  $(D - \lambda)x = e_1$  and thus  $Dx = \lambda x + e_1$ . This implies that

$$x(k-1) = Dx(k) = \lambda x(k) + e_1(k)$$

for all  $k \in \mathbb{N}$ . From this we have  $x(0) = -\lambda^{-1}$  and  $x(k) = -\lambda^{-k}x(0)$  for all  $k \geq 1$  because since  $e_1(0) = 1$  and  $e_1(k) = 0$  for all  $k \geq 1$ . Also since  $|\lambda| = 1$ , it turns out that  $|x(0)| = |x(k)|$  for all  $k \geq 1$ . As  $x \in \ell^2(\mathbb{N})$ , this forces  $x = 0$ . However, it is absurd because  $Dx = \lambda x + e_1$ .

Now we consider the case when  $|\lambda| < 1$ .

Notice that by Proposition 13.13, we have

$$\overline{\text{im}(D - \lambda)}^\perp = \ker(D - \lambda)^* = \ker(D^* - \bar{\lambda}).$$

Thus if  $D - \lambda$  is surjective, we have  $\ker(D^* - \bar{\lambda}) = (0)$  and hence  $\bar{\lambda} \notin \sigma_p(D^*)$ .

Notice that the adjoint  $D^*$  of  $D$  is given by the left shift operator, that is,

$$D^*x(k) = x(k+1) \quad \dots \dots \dots (**)$$

for all  $k \in \mathbb{N}$ .

Now when  $D^*x = \mu x$  for some  $\mu \in \mathbb{C}$  and  $x \in \ell^2(\mathbb{N})$ , by using Eq.(\*\*), which is equivalent to saying that

$$x(k+1) = \mu x(k)$$

for all  $k \in \mathbb{N}$ . So as  $|\bar{\lambda}| = |\lambda| < 1$ , if we set  $x(0) = 1$  and  $x(k+1) = \bar{\lambda}^k x(0)$  for all  $k \geq 1$ , then  $x \in \ell^2(\mathbb{N})$  and  $D^*x = \bar{\lambda}x$ . Hence  $\bar{\lambda} \in \sigma_p(D^*)$  which leads to a contradiction.

The proof is finished.

## 15. SPECTRAL THEORY II

Throughout this section, let  $H$  be a complex Hilbert space.

**Lemma 15.1.** *Let  $T \in B(H)$  be a normal operator (recall that  $T^*T = TT^*$ ). Then  $T$  is invertible in  $B(H)$  if and only if there is  $c > 0$  such that  $\|Tx\| \geq c\|x\|$  for all  $x \in H$ .*

*Proof.* The necessary part is clear.

Now we are going to show the converse. We first to show the case when  $T$  is selfadjoint. It is clear that  $T$  is injective from the assumption. So by the Open Mapping Theorem, it remains to show that  $T$  is surjective.

In fact since  $\ker T = \overline{\text{im}T^*}^\perp$  and  $T = T^*$ , we see that the image of  $T$  is dense in  $H$ .

Now if  $y \in H$ , then there is a sequence  $(x_n)$  in  $H$  such that  $Tx_n \rightarrow y$ . So  $(Tx_n)$  is a Cauchy sequence. From this and the assumption give us that  $(x_n)$  is also a Cauchy sequence. If  $x_n$  converges to  $x \in H$ , then  $y = Tx$ . Therefore the assertion is true when  $T$  is selfadjoint.

Now if  $T$  is normal, then we have  $\|T^*x\| = \|Tx\| \geq c\|x\|$  for all  $x \in H$  by Proposition 13.11(ii). Therefore, we have  $\|T^*Tx\| \geq c\|Tx\| \geq c^2\|x\|$ . Hence  $T^*T$  still satisfies the assumption. Notice that  $T^*T$  is selfadjoint. So we can apply the previous case to know that  $T^*T$  is invertible. This implies that  $T$  is also invertible because  $T^*T = TT^*$ .

The proof is finished. □

**Definition 15.2.** *Let  $T \in B(X)$ . We say that  $T$  is positive, write  $T \geq 0$ , if  $(Tx, x) \geq 0$  for all  $x \in H$ .*

**Remark 15.3.** *It is clear that a positive operator is selfadjoint by Proposition 13.12 at once. In particular, all projections are positive.*



**Proposition 15.4.** *Let  $T \in B(H)$ . We have*

- (i) : *If  $T \geq 0$ , then  $T + I$  is invertible.*
- (ii) : *If  $T$  is self-adjoint, then  $\sigma(T) \subseteq \mathbb{R}$ . In particular, when  $T \geq 0$ , we have  $\sigma(T) \subseteq [0, \infty)$ .*

*Proof.* For Part (i), we assume that  $T \geq 0$ . This implies that

$$\|(I + T)x\|^2 = \|x\|^2 + \|Tx\|^2 + 2(Tx, x) \geq \|x\|^2$$

for all  $x \in H$ . So the invertibility of  $I + T$  follows from Lemma 15.1.

For Part (ii), we first claim that  $T + i$  is invertible. Indeed, it follows from  $(T + i)^*(T + i) = T^2 + I$  and Part (i) immediately.

Now if  $\lambda = a + ib \in \sigma(T)$  where  $a, b \in \mathbb{R}$  with  $b \neq 0$ , then  $T - \lambda = -b(\frac{-1}{b}(T - a) + i)$  is invertible because  $\frac{-1}{b}(T - a)$  is selfadjoint.

Finally we are going to show  $\sigma(T) \subseteq [0, \infty)$  when  $T \geq 0$ . Notice that since  $\sigma(T) \subseteq \mathbb{R}$ , it suffices to show that  $T - c$  is invertible if  $c < 0$ . Indeed, if  $c < 0$ , then we see that  $T - c = -c(I + (\frac{-1}{c}T))$  is invertible by the previous assertion because  $\frac{-1}{c}T \geq 0$ .

The proof is finished.  $\square$

**Remark 15.5.** *In Proposition 15.4, we have shown that if  $T$  is selfadjoint, then  $\sigma(T) \subseteq \mathbb{R}$ . However, the converse does not hold. For example, consider  $H = \mathbb{C}^2$  and*

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

**Theorem 15.6.** *Let  $T \in B(H)$  be a selfadjoint operator. Put*

$$M(T) := \sup_{\|x\|=1} (Tx, x) \quad \text{and} \quad m(T) = \inf_{\|x\|=1} (Tx, x).$$

*For convenience, we also write  $M = M(T)$  and  $m = m(T)$  if there is no confusion.*

*Then we have*

- (i) :  $\|T\| = \max\{|m|, |M|\}$ .
- (ii) :  $\{m, M\} \subseteq \sigma(T)$ .
- (iii) :  $\sigma(T) \subseteq [m, M]$ .

*Proof.* Notice that  $m$  and  $M$  are defined because  $(Tx, x)$  is real for all  $x \in H$  by Proposition 13.12 (ii). Also Part(i) can be obtained by using Lemma 13.12 (ii) again.

For Part (ii), we first claim that  $M \in \sigma(T)$  if  $T \geq 0$ . Notice that  $0 \leq m \leq M = \|T\|$  in this case by Lemma 13.12. Then there is a sequence  $(x_n)$  in  $H$  with  $\|x_n\| = 1$  for all  $n$  such that  $(Tx_n, x_n) \rightarrow M = \|T\|$ . Then we have

$$\|(T - M)x_n\|^2 = \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \leq \|T\|^2 + M^2 - 2M(Tx_n, x_n) \rightarrow 0.$$

So by Lemma 15.1 we have shown that  $T - M$  is not invertible and hence  $M \in \sigma(T)$  if  $T \geq 0$ .

Now for any selfadjoint operator  $T$  if we consider  $T - m$ , then  $T - m \geq 0$ . Thus we have  $M - m = M(T - m) \in \sigma(T - m)$  by the previous case. It is clear that  $\sigma(T - c) = \sigma(T) - c$  for all  $c \in \mathbb{C}$ . Therefore we have  $M \in \sigma(T)$  for any self-adjoint operator.

We are now claiming that  $m(T) \in \sigma(T)$ . Notice that  $M(-T) = -m(T)$ . So we have  $-m(T) \in \sigma(-T)$ . It is clear that  $\sigma(-T) = -\sigma(T)$ . Then  $m(T) \in \sigma(T)$ .

Finally, we are going to show  $\sigma(T) \subseteq [m, M]$ .

Indeed, since  $T - m \geq 0$ , then by Proposition 15.4, we have  $\sigma(T) - m = \sigma(T - m) \subseteq [0, \infty)$ . This gives  $\sigma(T) \subseteq [m, \infty)$ .

On the other hand, similarly, we consider  $M - T \geq 0$ . Then we get  $M - \sigma(T) = \sigma(M - T) \subseteq [0, \infty)$ . This implies that  $\sigma(T) \subseteq (-\infty, M]$ . The proof is finished.  $\square$

## 16. COMPACT OPERATORS ON A HILBERT SPACE

Throughout this section, let  $H$  be a complex Hilbert space.

**Definition 16.1.** A linear operator  $T : H \rightarrow H$  is said to be compact if for every bounded sequence  $(x_n)$  in  $H$ ,  $(T(x_n))$  has a norm convergent subsequence.

Write  $K(H)$  for the set of all compact operators on  $H$  and  $K(H)_{sa}$  for the set of all compact selfadjoint operators.

**Remark 16.2.** Let  $U$  be the closed unit ball of  $H$ . It is clear that  $T$  is compact if and only if the norm closure  $\overline{T(U)}$  is a compact subset of  $H$ . Thus if  $T$  is compact, then  $T$  is bounded automatically because every compact set is bounded.

Also it is clear that if  $T$  has finite rank, that is  $\dim \operatorname{im} T < \infty$ , then  $T$  must be compact because every closed and bounded subset of a finite dimensional normed space is equivalent to it is compact.

**Example 16.3.** The identity operator  $I : H \rightarrow H$  is compact if and only if  $\dim H < \infty$ .

**Example 16.4.** Let  $H = \ell^2(\{1, 2, \dots\})$ . Define  $Tx(k) := \frac{x(k)}{k}$  for  $k = 1, 2, \dots$ . Then  $T$  is compact. In fact, if we let  $(x_n)$  be a bounded sequence in  $\ell^2$ , then by the diagonal argument, we can find a subsequence  $y_m := Tx_m$  of  $Tx_n$  such that  $\lim_{m \rightarrow \infty} y_m(k) = y(k)$  exists for all  $k = 1, 2, \dots$ . Let  $L := \sup_n \|x_n\|_2^2$ . Since  $|y_m(k)|^2 \leq \frac{L}{k^2}$  for all  $m, k$ , we have  $y \in \ell^2$ . Now let  $\varepsilon > 0$ . Then one can find a positive integer  $N$  such that  $\sum_{k \geq N} 4L/k^2 < \varepsilon$ . So we have

$$\sum_{k \geq N} |y_m(k) - y(k)|^2 < \sum_{k \geq N} \frac{4L}{k^2} < \varepsilon$$

for all  $m$ . On the other hand, since  $\lim_{m \rightarrow \infty} y_m(k) = y(k)$  for all  $k$ , we can choose a positive integer  $M$  such that

$$\sum_{k=1}^{N-1} |y_m(k) - y(k)|^2 < \varepsilon$$

for all  $m \geq M$ . Finally, we have  $\|y_m - y\|_2^2 < 2\varepsilon$  for all  $m \geq M$ .

**Theorem 16.5.** Let  $T \in B(H)$ . Then  $T$  is compact if and only if  $T$  maps every weakly convergent sequence in  $H$  to a norm convergent sequence.

*Proof.* We first assume that  $T \in K(H)$ . Let  $(x_n)$  be a weakly convergent sequence in  $H$ . Since  $H$  is reflexive,  $(x_n)$  is bounded by the Uniform Boundedness Theorem. So we can find a subsequence  $(x_j)$  of  $(x_n)$  such that  $(Tx_j)$  is norm convergent. Let  $y := \lim_j Tx_j$ . We claim that  $y = \lim_n Tx_n$ . Suppose not. Then by the compactness of  $T$  again, we can find a subsequence  $(x_i)$  of  $(x_n)$  such that  $Tx_i$  converges to  $y'$  with  $y \neq y'$ . Thus there is  $z \in H$  such that  $(y, z) \neq (y', z)$ . On the other hand, if we let  $x$  be the weakly limit of  $(x_n)$ , then  $(x_n, w) \rightarrow (x, w)$  for all  $w \in H$ . So we have

$$(y, z) = \lim_j (Tx_j, z) = \lim_j (x_j, T^*(z)) = (x, T^*(z)) = (Tx, z).$$

Similarly, we also have  $(y', z) = (Tx, z)$  and hence  $(y, z) = (y', z)$  that contradicts to the choice of  $z$ .

For the converse, let  $(x_n)$  be a bounded sequence. Then by Theorem 12.12,  $(x_n)$  has a weakly convergent subsequence. Thus  $T(x_n)$  has a norm convergent subsequence by the assumption at once. So  $T$  is compact.  $\square$

**Proposition 16.6.** *Let  $S, T \in K(H)$ . Then we have*

- (i) :  $\alpha S + \beta T \in K(H)$  for all  $\alpha, \beta \in \mathbb{C}$ ;
- (ii) :  $TQ$  and  $QT \in K(H)$  for all  $Q$  in  $B(H)$ ;
- (iii) :  $T^* \in K(H)$ .

Moreover  $K(H)$  is normed closed in  $B(H)$ .

Hence  $K(H)$  is a closed  $*$ -ideal of  $B(H)$ .

*Proof.* (i) and (ii) are clear.

For property (iii), let  $(x_n)$  be a bounded sequence. Then  $(T^*x_n)$  is also bounded. So  $TT^*x_n$  has a convergent subsequence  $TT^*x_{n_k}$  by the compactness of  $T$ . Notice that we have

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = (TT^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l})$$

for all  $n_k, n_l$ . This implies that  $(T^*x_{n_k})$  is a Cauchy sequence and thus is convergent since  $(x_{n_k})$  is bounded.

Finally we are going to show  $K(H)$  is closed. Let  $(T_m)$  be a sequence in  $K(H)$  such that  $T_m \rightarrow T$  in norm. Let  $(x_n)$  be a bounded sequence in  $H$ . Then by the diagonal argument there is a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $\lim_k T_m x_{n_k}$  exists for all  $m$ . Now let  $\varepsilon > 0$ . Since  $\lim_m T_m = T$ , there is a positive integer  $N$  such that  $\|T - T_N\| < \varepsilon$ . On the other hand, there is a positive integer  $K$  such that  $\|T_N x_{n_k} - T_N x_{n_{k'}}\| < \varepsilon$  for all  $k, k' \geq K$ . So we can now have

$$\|T x_{n_k} - T x_{n_{k'}}\| \leq \|T x_{n_k} - T_N x_{n_k}\| + \|T_N x_{n_k} - T_N x_{n_{k'}}\| + \|T_N x_{n_{k'}} - T x_{n_{k'}}\| \leq (2L + 1)\varepsilon$$

for all  $k, k' \geq K$  where  $L := \sup_n \|x_n\|$ . Thus  $\lim_k T x_{n_k}$  exists. It can now be concluded that  $T \in K(H)$ . The proof is finished.  $\square$

**Corollary 16.7.** *Let  $T \in K(H)$ . If  $\dim H = \infty$ , then  $0 \in \sigma(T)$ .*

*Proof.* Suppose that  $0 \notin \sigma(T)$ . Then  $T^{-1}$  exists in  $B(H)$ . Proposition 16.1 gives  $I = TT^{-1} \in K(H)$ . This implies  $\dim H < \infty$ .  $\square$

**Proposition 16.8.** *Let  $T \in K(H)$  and let  $c \in \mathbb{C}$  with  $c \neq 0$ . Then  $T - c$  has a closed range.*

*Proof.* Notice that since  $\frac{1}{c}T \in K(H)$ , so if we consider  $\frac{1}{c}T - I$ , we may assume that  $c = 1$ . Let  $S = T - I$ . Let  $x_n$  be a sequence in  $H$  such that  $Sx_n \rightarrow x \in H$  in norm. By considering the orthogonal decomposition  $H = \ker S \oplus (\ker S)^\perp$ , we write  $x_n = y_n \oplus z_n$  for  $y_n \in \ker S$  and  $z_n \in (\ker S)^\perp$ . We first claim that  $(z_n)$  is bounded. Suppose not. By considering a subsequence of  $(z_n)$ , we may assume that we may assume that  $\|z_n\| \rightarrow \infty$ . Put  $v_n := \frac{z_n}{\|z_n\|} \in (\ker S)^\perp$ .

Since  $Sz_n = Sx_n \rightarrow x$ , we have  $Sv_n \rightarrow 0$ . On the other hand, since  $T$  is compact, and  $(v_n)$  is bounded, by passing a subsequence of  $(v_n)$ , we may also assume that  $Tv_n \rightarrow w$ . Since  $S = T - I$ ,  $v_n = Tv_n - Sv_n \rightarrow w - 0 = w \in (\ker S)^\perp$ . Also from this we have  $Sv_n \rightarrow Sw$ . On the other hand, we have  $Sw = \lim_n Sv_n = \lim_n Tv_n - \lim_n v_n = w - w = 0$ . So  $w \in \ker S \cap (\ker S)^\perp$ . It follows that  $w = 0$ . However, since  $v_n \rightarrow w$  and  $\|v_n\| = 1$  for all  $n$ . It leads to a contradiction. So  $(z_n)$  is bounded.

Finally we are going to show that  $x \in \text{im} S$ . Now since  $(z_n)$  is bounded,  $(Tz_n)$  has a convergent subsequence  $(Tz_{n_k})$ . Let  $\lim_k Tz_{n_k} = z$ . Then we have

$$z_{n_k} = Sz_{n_k} - Tz_{n_k} = Sx_{n_k} - Tz_{n_k} \rightarrow x - z.$$

It follows that  $x = \lim_k Sx_{n_k} = \lim_k Sz_{n_k} = S(x - z) \in \text{im} S$ . The proof is finished.  $\square$

**Theorem 16.9. Fredholm Alternative Theorem :** *Let  $T \in K(H)_{sa}$  and let  $0 \neq \lambda \in \mathbb{C}$ . Then  $T - \lambda$  is injective if and only if  $T - \lambda$  is surjective.*

*Proof.* Since  $T$  is selfadjoint,  $\sigma(T) \subseteq \mathbb{R}$ . So if  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , then  $T - \lambda$  is invertible. So the result holds automatically.

Now consider the case  $\lambda \in \mathbb{R} \setminus \{0\}$ .

Then  $T - \lambda$  is also selfadjoint. From this and Proposition 13.13, we have  $\ker(T - \lambda) = (\text{im}(T - \lambda))^\perp$  and  $(\ker(T - \lambda))^\perp = \overline{\text{im}(T - \lambda)}$ .

So the proof is finished by using Proposition 16.8 immediately.  $\square$

**Corollary 16.10.** *Let  $T \in K(H)_{sa}$ . Then we have  $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$ . Consequently if the values  $m(T)$  and  $M(T)$  which are defined in Theorem 15.6 are non-zero, then both are the eigenvalues of  $T$  and  $\|T\| = \max_{\lambda \in \sigma_p(T)} |\lambda|$ .*

*Proof.* It follows from the Fredholm Alternative Theorem at once. This together with Theorem 15.6 imply the last assertion.  $\square$

**Example 16.11.** *Let  $T \in B(\ell^2)$  be defined as in Example 16.4. We have shown that  $T \in K(\ell^2)$  and it is clear that  $T$  is selfadjoint. Then by Corollary 16.10 and Corollary 16.7, we see that  $\sigma(T) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ .*

**Lemma 16.12.** *Let  $T \in K(H)_{sa}$  and let  $E_\lambda := \{x \in H : Tx = \lambda x\}$  for  $\lambda \in \sigma(T) \setminus \{0\}$ , that is the eigenspace of  $T$  corresponding to  $\lambda$ . If we fix  $\mu \in \sigma(T) \setminus \{0\}$  and put  $I_\mu := \{\lambda \in \sigma(T) : |\lambda| = |\mu|\}$ , then we have*

$$\dim \bigoplus_{\lambda \in I_\mu} E_\lambda < \infty.$$

*Proof.* We first notice that  $\dim E_\lambda < \infty$  for all  $\lambda \in \sigma_p(T) \setminus \{0\}$  because the restriction  $T|_{E_\lambda}$  is also a compact operator on  $E_\lambda$ .

On the other hand, since  $T$  is selfadjoint, we also have  $E_\lambda \perp E_{\lambda'}$  for  $\lambda, \lambda' \in \sigma_p(T)$  with  $\lambda \neq \lambda'$ . Let  $V := \bigoplus_{\lambda \in I_\mu} E_\lambda$ . Suppose that  $\dim V = \infty$ . Then  $|I_\mu| = \infty$ . So, we can find an infinite sequence in  $I_\mu$  such that  $\lambda_m \neq \lambda_n$  for  $m \neq n$ . Now choose  $v_n \in E_{\lambda_n}$  with  $\|v_n\| = 1$  for each  $\lambda_n$ . Then  $v_n \perp v_m$  for  $n \neq m$ . This implies that  $\|Tv_n - Tv_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 = 2|\mu|^2 > 0$  for  $m \neq n$ . So  $(Tv_n)$  has no convergent subsequences which contradicts to  $T$  being compact.  $\square$

**Theorem 16.13.** *Let  $T \in K(H)_{sa}$ . And suppose that  $\dim H = \infty$ . Then  $\sigma(T) = \{\lambda_1, \lambda_2, \dots\} \cup \{0\}$ , where  $(\lambda_n)$  is a sequence of real numbers with  $\lambda_n \neq \lambda_m$  for  $m \neq n$  and  $|\lambda_n| \downarrow 0$ .*

*Proof.* Note that since  $\|T\| = \max(|M(T)|, |m(T)|)$  and  $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$ . So by Corollary 16.10, there is  $|\lambda_1| = \max_{\lambda \in \sigma_p(T)} |\lambda| = \|T\|$ . Since  $\dim E_{\lambda_1} < \infty$ , then  $E_{\lambda_1}^\perp \neq 0$ . Then by considering

the restriction of  $T_2 := T|_{E_{\lambda_1}^\perp} \neq 0$ , there is  $|\lambda_2| = \max_{\lambda \in \sigma_p(T_2)} |\lambda| = \|T_2\|$ . Notice that  $\lambda_2 \in \sigma_p(T)$  and  $|\lambda_2| \leq |\lambda_1|$  because  $\|T_2\| \leq \|T\|$ . To repeat the same step, we can get a sequence  $(\lambda_n)$  such that  $(|\lambda_n|)$  is decreasing.

Now we claim that  $\lim_n |\lambda_n| = 0$ .

Otherwise, there is  $\eta > 0$  such that  $|\lambda_n| \geq \eta$  for all  $n$ . If we let  $v_n \in E_{\lambda_n}$  with  $\|v_n\| = 1$  for all  $n$ . Notice that since  $\dim H = \infty$  and  $\dim E_\lambda < \infty$ , for any  $\lambda \in \sigma_p(T) \setminus \{0\}$ , there are infinite many  $\lambda_n$ 's. Then  $w_n := \frac{1}{|\lambda_n|} v_n$  is a bounded sequence and  $\|Tw_n - Tw_m\|^2 = \|v_n - v_m\|^2 = 2$  for  $m \neq n$ . This is a contradiction since  $T$  is compact. So  $\lim_n |\lambda_n| = 0$ .

Finally we need to check  $\sigma(T) = \{\lambda_1, \lambda_2, \dots\} \cup \{0\}$ .

In fact, let  $\mu \in \sigma_p(T)$ . Since  $|\lambda_n| \downarrow 0$ , we can find a subsequence  $n_1 < n_2 < \dots$  of positive integers such that

$$|\lambda_1| = \dots = |\lambda_{n_1}| > |\lambda_{n_1+1}| = \dots = |\lambda_{n_2}| > |\lambda_{n_2+1}| = \dots = |\lambda_{n_3}| > |\lambda_{n_3+1}| = \dots$$

Then we can choose  $N$  such that  $|\lambda_{n_{N+1}}| < |\mu| \leq |\lambda_{n_N}|$ . Notice that by the construction of  $\lambda_n$ 's implies  $\mu = \lambda_j$  for some  $n_{N-1} + 1 \leq j \leq n_N$ .

The proof is finished.  $\square$

**Theorem 16.14.** *Let  $T \in K(H)_{sa}$  and let  $(\lambda_n)$  be given as in Theorem 16.13. For each  $\lambda \in \sigma_p(T) \setminus \{0\}$ , put  $d(\lambda) := \dim E_\lambda < \infty$ . Let  $\{e_{\lambda,i} : i = 1, \dots, d(\lambda)\}$  be an orthonormal base for  $E_\lambda$ . Then we have the following orthogonal decomposition:*

$$(16.1) \quad H = \ker T \oplus \bigoplus_{n=1}^{\infty} E_{\lambda_n}.$$

Moreover  $\mathcal{B} := \{e_{\lambda,i} : \lambda \in \sigma_p(T) \setminus \{0\}; i = 1, \dots, d(\lambda)\}$  forms an orthonormal base of  $\overline{T(H)}$ .

Also the series  $\sum_{n=1}^{\infty} \lambda_n P_n$  norm converges to  $T$ , where  $P_n$  is the orthogonal projection from  $H$  onto

$$E_{\lambda_n}, \text{ that is, } P_n(x) := \sum_{i=1}^{d(\lambda_n)} (x, e_{\lambda_n,i}) e_{\lambda_n,i}, \text{ for } x \in H.$$

*Proof.* Put  $E = \bigoplus_{n=1}^{\infty} E_{\lambda_n}$ . It is clear that  $\ker T \subseteq E^\perp$ . On the other hand, if the restriction  $T_0 := T|_{E^\perp} \neq 0$ , then there exists a non-zero element  $\mu \in \sigma_p(T_0) \subseteq \sigma_p(T)$  because  $T_0 \in K(E^\perp)$ . It is absurd because  $\mu \neq \frac{1}{\lambda_i}$  for all  $i$ . So  $T|_{E^\perp} = 0$  and hence  $E^\perp \subseteq \ker T$ . So we have the decomposition (16.1). And from this we see that the family  $\mathcal{B}$  forms an orthonormal base of  $(\ker T)^\perp$ . On the other, we have  $(\ker T)^\perp = \overline{im T^*} = \overline{im T}$ . Therefore,  $\mathcal{B}$  is an orthonormal base for  $\overline{T(H)}$  as desired.

For the last assertion, it needs to show that the series  $\sum_{n=1}^{\infty} \lambda_n P_n$  converges to  $T$  in norm. Notice that if we put  $S_m := \sum_{n=1}^m \lambda_n P_n$ , then by the decomposition (16.1),  $\lim_{m \rightarrow \infty} S_m x = T x$  for all  $x \in H$ . So it suffices to show that  $(S_m)_{m=1}^{\infty}$  is a Cauchy sequence in  $B(H)$ . In fact we have

$$\|\lambda_{m+1} P_{m+1} + \dots + \lambda_{m+p} P_{m+p}\| = |\lambda_{m+1}|$$

for all  $m, p \in \mathbb{N}$  because  $E_{\lambda_n} \perp E_{\lambda_m}$  for  $m \neq n$  and  $|\lambda_n|$  is decreasing. This gives that  $(S_m)$  is a Cauchy sequence since  $|\lambda_n| \downarrow 0$ . The proof is finished.  $\square$

**Corollary 16.15.**  *$T \in K(H)$  if and only if  $T$  can be approximated by finite rank operators.*

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